FINITE-DIMENSIONAL STRUCTURE AND BEST APPROXIMATION IN NEUTROSOPHIC INTUITIONISTIC ALGEBRA FUZZY NORMED SPACES

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Abstract. This paper introduces and studies the finite-dimensional structure of neutrosophic intuitionistic algebra fuzzy normed spaces. The proposed framework is described by three independent functions representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, together with a continuous t-norm and a continuous t-conorm. Fundamental algebraic and topological notions are developed, including convergence, Cauchy sequences, completeness, boundedness, open and closed balls, neighbourhoods, closure, density, limit points, total boundedness, and compactness. We establish that every convergent sequence is Cauchy and investigate the relationships among compactness, sequential compactness, total boundedness, and completeness. In particular, compactness is shown to be equivalent to total boundedness together with completeness under the assumed neutrosophic intuitionistic algebra fuzzy structure. We also introduce an appropriate notion of equivalence for norm-compatible neutrosophic intuitionistic algebra fuzzy norms and prove that any two such norms on a finite-dimensional vector space generate the same convergence and topological structure. As a consequence, a Heine–Borel type theorem is obtained: a subset of a finite-dimensional neutrosophic intuitionistic algebra fuzzy normed space is compact if and only if it is closed and bounded. Finally, a Riesz-type separation lemma is established and used to show that the compactness of a nontrivial closed neutrosophic intuitionistic algebra fuzzy ball characterizes finite-dimensionality. As an application, a neutrosophic best-approximation theorem is obtained: every nonempty closed and bounded subset admits a weighted best approximant at each positive scale. The resulting approximant is Pareto efficient with respect to the truth, indeterminacy, and falsity errors, and it is unique under an appropriate strict-convexity assumption.

1. Introduction

Classical normed-space theory provides a precise setting for studying convergence, completeness, boundedness, compactness, and continuous linear operators. Its binary description of membership, however, is often too rigid for mathematical models involving vagueness, incomplete information, or indeterminacy. Zadeh's theory of fuzzy sets initiated a systematic treatment of graded membership [19], while Atanassov's intuitionistic fuzzy sets supplemented membership by a non-membership degree and the resulting hesitation margin [20]. The relationships among such extensions of fuzzy-set theory were subsequently clarified in [21]. In parallel, fuzzy compactness and fuzzy topological structures were developed in [22], and the interaction between fuzziness and linear structure was investigated through fuzzy vector spaces

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and fuzzy topological vector spaces [23, 24]. These developments established the topological and algebraic background needed for fuzzy analogues of functional analysis.

The passage from fuzzy topology to fuzzy normed spaces led to several formulations of a fuzzy norm. Felbin's finite-dimensional theory showed that many central features of ordinary finite-dimensional normed spaces admit meaningful fuzzy counterparts [5]. The metrically induced approach of George and Veeramani supplied useful Hausdorff topologies and analytic tools for fuzzy metric spaces [25, 26]. Further work examined the topological structure generated by fuzzy norms [16] and established compactness results in L-fuzzy Euclidean normed spaces [4]. Thus, notions such as balls, neighbourhoods, closure, boundedness, completeness, and compactness became available in a form compatible with graded uncertainty.

Intuitionistic fuzzy normed spaces refined this framework by describing both membership and non-membership through a continuous t-norm and a continuous t-conorm. Park's intuitionistic fuzzy metric spaces [27] and the intuitionistic fuzzy topological and normed structures studied by Saadati and Park [15] provided an influential foundation. Related properties of intuitionistic fuzzy normed spaces were investigated by Lael and Nourouzi [28], while later analyses clarified the relation between intuitionistic fuzzy metric and normed formulations [29]. Modified intuitionistic fuzzy metric spaces were developed in [30], and the theory was extended to ntuitionistic 2-fuzzy n-normed spaces [31],intuitionistic fuzzy topological vector spaces [32], and intuitionistic type-2 fuzzy normed linear spaces [33]. These studies demonstrate that the interaction among algebraic operations, fuzzy norms, and induced topologies is both substantial and sensitive to the chosen axioms. These studies demonstrate that the interaction among algebraic operations, fuzzy norms, and induced topologies is both substantial and sensitive to the chosen axioms [35, 34].

Another major line of research concerns generalized convergence in intuitionistic fuzzy normed spaces. Statistical convergence was studied in [6], followed by generalized statistical convergence [36], lacunary statistical convergence [37], and ideal convergence of double sequences [38]. Generalized ideal convergence was further developed in [39]. Cesàro and logarithmic summability, together with associated Tauberian conditions, were treated in [40, 41], whereas rough statistical convergence of double sequences was considered in [42]. Although these results greatly enrich sequence theory, they mainly address particular convergence or summability methods rather than a unified finite-dimensional theory of topology and compactness.

Neutrosophic analysis offers a broader representation of uncertainty by using three components for truth, indeterminacy, and falsity [43, 44, 45]. In contrast with an intuitionistic fuzzy pair, these components are treated separately and need not be determined by one another. Kirişci and Şimşek formulated neutrosophic normed spaces and investigated statistical convergence and statistical completeness in that setting [11], and Ahmad and Mursaleen [12]. Subsequent work developed continuous and bounded linear operators [9], nonlinear operators and Fréchet differentiation [10], and neutrosophic Tribonacci sequence spaces [7, 8]. The definition and convergence theory were further examined through rough lacunary statistical convergence [46, 47, 48, 49], lacunary ideal convergence [50], and deferred statistical convergence of double sequences [51, 52]. Recent investigations have also considered stability of functional equations [53], bounded and compact linear operators [54], and convergence involving neutrosophic operators [55]. These contributions confirm the potential of the three-component norm, but they also show that the basic topological consequences of its axioms must be established carefully before more specialized operator or sequence theories are developed [56].

Finite-dimensional spaces provide the natural testing ground for this foundational task. In classical analysis, equivalence of norms, completeness of finite-dimensional spaces, the Heine–Borel theorem, and the compactness of closed balls form a tightly connected group of results. Related questions have been revisited for the completion of fuzzy normed linear spaces [57], for the comparison between Felbin and Bag–Samanta fuzzy norms [58], and for finite-dimensional fuzzy strong ϕ-b-normed linear spaces [59]. Nevertheless, the literature reviewed above does

not provide a single systematic treatment that derives the corresponding finite-dimensional equivalence, compactness, and separation results for a norm-compatible three-component neutrosophic intuitionistic algebra fuzzy structure. This is the gap addressed in the present paper.

Accordingly, we introduce and study finite-dimensional neutrosophic intuitionistic algebra fuzzy normed spaces described by three functions representing truth-membership, indeterminacy-membership, and falsity-membership degrees, together with a continuous t-norm and a continuous t-conorm. We first develop the associated notions of convergence, Cauchy sequence, completeness, boundedness, open and closed balls, neighbourhood, closure, density, limit point, total boundedness, and compactness. We prove that convergence implies the Cauchy property and establish the relevant implications among compactness, sequential compactness, total boundedness, and completeness. In particular, compactness is characterized by total boundedness and completeness under the adopted neutrosophic intuitionistic algebra fuzzy axioms. We then define an equivalence relation for norm-compatible neutrosophic intuitionistic algebra fuzzy norms and prove that any two such norms on a finite-dimensional vector space induce the same convergence and topology. This yields a Heine–Borel type theorem: a subset is compact if and only if it is closed and bounded. Finally, we establish a Riesz-type separation lemma and use it to prove that a nontrivial closed neutrosophic intuitionistic algebra fuzzy ball is compact exactly in the finite-dimensional case. As an application, every nonempty closed and bounded subset admits a Pareto-efficient weighted neutrosophic best approximant at each positive scale, which is unique when the set is convex and the approximation functional is strictly convex. Together, these results provide a coherent finite-dimensional foundation for subsequent work on operators, approximation, and functional equations in this three-component setting.

2. Literature Review

2.1. **Finite-dimensional fuzzy normed spaces.** Several formulations of fuzzy normed linear spaces have been proposed, and their finite-dimensional properties depend on the adopted definition of a fuzzy norm. Felbin [5] introduced a fuzzy norm whose value is a fuzzy real number and developed fundamental results for finite-dimensional spaces. Bag and Samanta [1] studied finite-dimensional fuzzy normed linear spaces under a different fuzzy-norm formulation and established analogues of several classical results concerning convergence, completeness, and linear structure. They also developed a theory of fuzzy bounded linear operators [2].

Compactness in finite-dimensional fuzzy normed spaces was investigated by Bag and Samanta [3], who obtained a fuzzy analogue of the classical result that closed and bounded subsets of finite-dimensional normed spaces are compact. In the related setting of L-fuzzy Euclidean normed spaces, Deschrijver et al. [4] studied compact subsets and established corresponding compactness criteria. These studies demonstrate that Heine–Borel type results can be recovered in particular fuzzy-norm frameworks. Nevertheless, their proofs depend on the specific structure of the underlying fuzzy norm and therefore cannot be transferred automatically to a norm described by three independent components.

The topology generated by a fuzzy norm is another important issue. Sadeqi and Kia [16] investigated the topological structure of fuzzy normed linear spaces, including the relationships among fuzzy convergence, neighbourhoods, and continuity. The completion problem for Bag–Samanta fuzzy normed linear spaces was subsequently studied by Yan [57]. Yaqub Azari and Sadeqi [58] showed that, under appropriate assumptions, Felbin-type and Bag–Samanta-type fuzzy norms generate equivalent topologies. Their result explains how some topological properties may be transferred between two established fuzzy-norm models. It does not, however, address norm equivalence in a framework involving independent truth, indeterminacy, and falsity functions.

2.2. **Intuitionistic fuzzy normed structures.** Intuitionistic fuzzy normed spaces extend fuzzy normed spaces by employing membership and non-membership functions. Their topological basis is connected with the intuitionistic fuzzy metric and topological structures studied by Park [27] and Saadati and Park [15]. Lael and Nourouzi [28] developed fundamental results for intuitionistic fuzzy normed spaces, while Mursaleen and Danish Lohani [13] extended the theory to intuitionistic fuzzy 2-normed spaces. Li, Guo, and Su [29] further examined the relationships between intuitionistic fuzzy metric spaces and intuitionistic fuzzy normed spaces.

Topological questions have also been considered in more general intuitionistic fuzzy linear structures. Chiney and Samanta [32] studied intuitionistic fuzzy topological vector spaces, and Kočinac et al. [31] investigated topological properties of intuitionistic 2-fuzzy n-normed linear spaces. More recently, Biswas, Chiney, and Samanta [33] introduced intuitionistic type-2 fuzzy normed linear spaces and established some of their basic properties. These contributions broaden the available intuitionistic fuzzy structures, but a unified treatment of finite-dimensional compactness, norm equivalence, and compact-ball characterizations is not provided.

A substantial part of the literature on intuitionistic fuzzy normed spaces is devoted to generalized convergence. Karakuş, Demirci, and Duman [6] introduced statistical convergence in intuitionistic fuzzy normed spaces. This work was extended to generalized statistical convergence [36], lacunary statistical convergence [37], and ideal convergence of double sequences [38]. Hazarika, Kumar, and Lafuerza-Guillén [39] considered generalized ideal convergence, whereas Talo and Yavuz [40] studied Cesàro summability and related Tauberian theorems. These results provide refined sequence-convergence methods, but their principal objectives differ from the finite-dimensional topological questions considered in the present study.

2.3. **Neutrosophic normed spaces.** In a neutrosophic framework, truth, indeterminacy, and falsity are treated as independent components [61]. This differs from an intuitionistic fuzzy structure, where the hesitation degree is determined by the membership and non-membership values. Consequently, topological results from intuitionistic fuzzy normed spaces require separate verification in a neutrosophic setting.

Kirişci and Şimşek [11] introduced neutrosophic normed spaces and studied statistical convergence in these spaces. Subsequent investigations have focused predominantly on generalized convergence and operator theory. Khan et al. [9] examined continuous and bounded linear operators between neutrosophic normed spaces and proved the continuity of vector addition and scalar multiplication. Khan and Khan [10] studied nonlinear operators, neutrosophic continuity, and Fréchet differentiation.

Several generalized convergence methods have also been transferred to neutrosophic normed spaces. Rough lacunary statistical convergence was investigated by Khan et al. [46], while Kişi and Choudhury [51] considered deferred statistical convergence of double sequences. Chaurasiya et al. [54] studied Cauchy sequences and different classes of bounded and compact linear operators, and Khan and Arshad [55] examined the convergence of sequences involving neutrosophic operators. Although these works substantially develop neutrosophic convergence and operator theory, they do not provide a complete theory connecting finite-dimensionality, compactness, total boundedness, completeness, and equivalence of norms.

2.4. **Compactness and finite-dimensionality.** In classical functional analysis, finite-dimensionality is closely related to norm equivalence, the Heine–Borel theorem, and the compactness of closed balls. Fuzzy analogues of some of these results are known in specific settings. Bag and Samanta [3] obtained compactness results for finite-dimensional fuzzy normed spaces, and Deschrijver et al. [4] treated compactness in L-fuzzy Euclidean normed spaces. Das and Bag [59] studied finite-dimensional fuzzy strong ϕ-b-normed linear spaces and extended a Riesz-type lemma to that setting.

2.5. **Best Approximation in Fuzzy Normed Structures.** Best approximation in fuzzy normed spaces was studied by Vaezpour and Karimi [62] through t-best approximants and t-proximinal sets. This theory was extended to simultaneous approximation by Goudarzi and Vaezpour [63] and to intuitionistic fuzzy normed spaces by Kavikumar et al. [64]. Related existence and nearest-point results were obtained in fuzzy metric and fuzzy quasi-normed spaces [65, 66]. These approaches use either one membership function or a membership–non-membership pair. In contrast, the present paper introduces a weighted approximation involving the truth, indeterminacy, and falsity errors and establishes existence, Pareto efficiency, and uniqueness under suitable compactness and convexity conditions.

These results are important precedents, but they rely on one- or two-component fuzzy norms. In particular, they do not simultaneously accommodate three independent functions representing truth-membership, indeterminacy-membership, and falsity-membership. The classical implication

$$\text{finite-dimensionality} \quad \Longleftrightarrow \quad \text{compactness of a nontrivial closed ball}$$

has therefore not been established for the neutrosophic intuitionistic algebra fuzzy structure considered here.

2.6. **Research gap.** The existing literature may be summarized in three principal directions. First, finite-dimensional fuzzy normed spaces provide results concerning compactness, boundedness, and induced topologies. Second, intuitionistic fuzzy normed spaces extend the underlying uncertainty model and support a developed theory of generalized convergence. Third, neutrosophic normed spaces provide independent truth, indeterminacy, and falsity components, but their existing theory focuses primarily on statistical convergence and operators.

What remains missing is a unified finite-dimensional theory for a neutrosophic intuitionistic algebra fuzzy norm that simultaneously addresses:

(1) convergence, Cauchy sequences, and completeness;
(2) boundedness, total boundedness, and compactness;
(3) the equivalence of compatible norms in finite dimensions;
(4) invariance of the induced convergence and topology;
(5) a Heine–Borel type characterization of compact subsets;
(6) a Riesz-type separation result characterizing finite-dimensionality through compact closed balls; and
(7) the existence, Pareto efficiency, and uniqueness conditions for weighted neutrosophic best approximants.

The present study is positioned at this intersection. It extends the finite-dimensional compactness theory of fuzzy normed spaces to a three-component neutrosophic intuitionistic algebra fuzzy setting and establishes the relationships among compactness, total boundedness, completeness, norm equivalence, and finite-dimensionality.

3. Basic Properties of Neutrosophic Intuitionistic Algebra Fuzzy Normed Spaces

In this section, we present the fundamental concepts required for the study of neutrosophic intuitionistic algebra fuzzy normed spaces. We first recall single-valued neutrosophic sets, continuous t-norms, and continuous t-conorms. We then introduce neutrosophic intuitionistic algebra fuzzy absolute values and neutrosophic intuitionistic algebra fuzzy norms.

Furthermore, we discuss convergence, Cauchy sequences, boundedness, completeness, neutrosophic intuitionistic fuzzy neighbourhoods, open and closed balls, interior points, adherent points, closure, and density. These notions provide the algebraic and topological framework needed for the subsequent investigation of compactness, completeness, finite-dimensional spaces, and the equivalence of neutrosophic intuitionistic algebra fuzzy norms.

3.1. **Neutrosophic Sets, t-Norms, and t-Conorms.** We begin with the basic notion of a single-valued neutrosophic set.

Definition 3.1. *Let S be a nonempty set. A* single-valued neutrosophic set $\widetilde{D}$ *in S is represented by*

$$\widetilde{D} = \left\{ \left(s, \Omega_{\widetilde{D}}(s), \Upsilon_{\widetilde{D}}(s), \Theta_{\widetilde{D}}(s) \right) : s \in S \right\},$$

where

$$\Omega_{\widetilde{D}}, \Upsilon_{\widetilde{D}}, \Theta_{\widetilde{D}} : S \longrightarrow [0,1]$$

are called the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively. For every $s \in S$, they satisfy

$$0 \leq \Omega_{\widetilde{D}}(s) + \Upsilon_{\widetilde{D}}(s) + \Theta_{\widetilde{D}}(s) \leq 3.$$

The three components in Definition 3.1 are independent. In particular, they are not required to have sum equal to one. The quantity

$$\sigma_{\widetilde{D}}(s) = \Omega_{\widetilde{D}}(s) + \Upsilon_{\widetilde{D}}(s) + \Theta_{\widetilde{D}}(s)$$

is called the total neutrosophic information degree of s with respect to $\widetilde{D}$.

Definition 3.2. *A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous t-norm if, for all $a, b, c, d \in [0,1]$, the following conditions hold:*

(1) $a * b = b * a$;
(2) $a * (b * c) = (a * b) * c$;
(3) $*$ *is continuous;*
(4) $a * 1 = a$;
(5) *if $a \leq c$ and $b \leq d$, then $a * b \leq c * d$.*

Example 3.1. *The product operation $a * b = ab, \qquad a, b \in [0,1]$, is a continuous t-norm.*

Example 3.2. *The minimum operation $a * b = \min\{a, b\}, \qquad a, b \in [0,1]$, is a continuous t-norm.*

Definition 3.3. *A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous t-conorm if, for all $a, b, c, d \in [0,1]$, the following conditions hold:*

(1) $a \diamond b = b \diamond a$;
(2) $a \diamond (b \diamond c) = (a \diamond b) \diamond c$;
(3) $\diamond$ *is continuous;*
(4) $a \diamond 0 = a$;
(5) *if $a \leq c$ and $b \leq d$, then $a \diamond b \leq c \diamond d$.*

Example 3.3. *The algebraic sum $a \diamond b = a + b - ab, \qquad a, b \in [0,1]$, is a continuous t-conorm.*

Example 3.4. *The maximum operation $a \diamond b = \max\{a, b\}, \qquad a, b \in [0,1]$, is a continuous t-conorm.*

Lemma 3.1. *Let $\diamond$ be a continuous t-conorm on $[0,1]$. Then*

$$1 \diamond 1 = 1, \qquad 0 \diamond 1 = 1 \diamond 0 = 1, \qquad 0 \diamond 0 = 0.$$

Moreover, $a \diamond a \geq a$ for every $a \in [0,1]$.

Proof. Using the identity property, we have $1 \diamond 0 = 1$. By commutativity, $0 \diamond 1 = 1$. Since $\diamond$ takes values in $[0,1]$ and is nondecreasing,

$$1 = 1 \diamond 0 \leq 1 \diamond 1 \leq 1.$$

Thus, $1 \diamond 1 = 1$. Also, $0 \diamond 0 = 0$. Finally, since $0 \leq a$, monotonicity gives $a = a \diamond 0 \leq a \diamond a$. □

Remark 3.1. *Let $\diamond$ be a continuous t-conorm.*

(1) *If* $a, b \in (0,1)$ *and* $a > b$, *then there exists* $w \in (0,1)$ *such that* $a > b \diamond w$. *Indeed,* $b \diamond 0 = b < a$, *and the assertion follows from continuity.*
(2) *For every* $\varepsilon \in (0,1)$, *there exists* $\delta \in (0,1)$ *such that* $\delta \diamond \delta < \varepsilon$. *This follows from continuity at* $(0,0)$ *and* $0 \diamond 0 = 0$.

Remark 3.2. *Let* $*$ *be a continuous t-norm. For every* $\varepsilon \in (0,1)$, *there exists* $\delta \in (0,1)$ *such that*

$$(1-\delta) * (1-\delta) > 1 - \varepsilon.$$

Indeed, the assertion follows from continuity at $(1,1)$ *and* $1 * 1 = 1$.

Remark 3.3. *In the present three-component framework, the continuous t-norm* $*$ *controls lower estimates for truth-membership values, whereas the continuous t-conorm* $\diamond$ *controls upper estimates for both indeterminacy-membership and falsity-membership values. This convention ensures that elements become closer to zero when their truth-membership approaches one and their indeterminacy- and falsity-membership values approach zero.*

3.2. **Neutrosophic Intuitionistic Algebra Fuzzy Absolute Value Spaces.** We next introduce the scalar structure required in the definition of a neutrosophic intuitionistic algebra fuzzy normed space.

Definition 3.4. *Let* $*$ *be a continuous t-norm and let* $\diamond$ *be a continuous t-conorm. A triple* $(\Omega_{\mathbb{R}}, \Upsilon_{\mathbb{R}}, \Theta_{\mathbb{R}})$ *of mappings from* $\mathbb{R}$ *into* $[0,1]$ *is called a* neutrosophic intuitionistic algebra fuzzy absolute value *on* $\mathbb{R}$ *if, for all* $\alpha, \beta \in \mathbb{R}$, *the following conditions hold:*

(1)
$$0 \le \Omega_{\mathbb{R}}(\alpha), \Upsilon_{\mathbb{R}}(\alpha), \Theta_{\mathbb{R}}(\alpha) \le 1;$$
(2) *The mappings* $\Omega_{\mathbb{R}}$, $\Upsilon_{\mathbb{R}}$, *and* $\Theta_{\mathbb{R}}$ *are continuous;*
(3) $\Omega_{\mathbb{R}}(\alpha) = 1 \Longleftrightarrow \alpha = 0$;
(4) $\Upsilon_{\mathbb{R}}(\alpha) = 0 \Longleftrightarrow \alpha = 0$;
(5) $\Theta_{\mathbb{R}}(\alpha) = 0 \Longleftrightarrow \alpha = 0$;
(6)
$$\Omega_{\mathbb{R}}(-\alpha) = \Omega_{\mathbb{R}}(\alpha), \quad \Upsilon_{\mathbb{R}}(-\alpha) = \Upsilon_{\mathbb{R}}(\alpha), \quad \Theta_{\mathbb{R}}(-\alpha) = \Theta_{\mathbb{R}}(\alpha);$$
(7)
$$\Omega_{\mathbb{R}}(\alpha\beta) \ge \Omega_{\mathbb{R}}(\alpha) * \Omega_{\mathbb{R}}(\beta);$$
(8)
$$\Upsilon_{\mathbb{R}}(\alpha\beta) \le \Upsilon_{\mathbb{R}}(\alpha) \diamond \Upsilon_{\mathbb{R}}(\beta);$$
(9)
$$\Theta_{\mathbb{R}}(\alpha\beta) \le \Theta_{\mathbb{R}}(\alpha) \diamond \Theta_{\mathbb{R}}(\beta);$$
(10)
$$\Omega_{\mathbb{R}}(\alpha+\beta) \ge \Omega_{\mathbb{R}}(\alpha) * \Omega_{\mathbb{R}}(\beta);$$
(11)
$$\Upsilon_{\mathbb{R}}(\alpha+\beta) \le \Upsilon_{\mathbb{R}}(\alpha) \diamond \Upsilon_{\mathbb{R}}(\beta);$$
(12)
$$\Theta_{\mathbb{R}}(\alpha+\beta) \le \Theta_{\mathbb{R}}(\alpha) \diamond \Theta_{\mathbb{R}}(\beta).$$

The system $(\mathbb{R}, \Omega_{\mathbb{R}}, \Upsilon_{\mathbb{R}}, \Theta_{\mathbb{R}}, *, \diamond)$ *is called a* neutrosophic intuitionistic algebra fuzzy absolute value space*, abbreviated as an* NIAFAVS.

Example 3.5. *Let* $\lambda \in (0,1)$, *let* $a * b = ab$ *and* $a \diamond b = a + b - ab$ *for all* $a, b \in [0,1]$. *For* $\alpha \in \mathbb{R}$, *define as* $q(\alpha) = \frac{|\alpha|}{1+|\alpha|}$, *and*

$$\Omega_{\mathbb{R}}(\alpha) = \frac{1}{1+|\alpha|}, \qquad \Upsilon_{\mathbb{R}}(\alpha) = \lambda q(\alpha), \qquad \Theta_{\mathbb{R}}(\alpha) = (1-\lambda) q(\alpha).$$

Then, $(\mathbb{R}, \Omega_{\mathbb{R}}, \Upsilon_{\mathbb{R}}, \Theta_{\mathbb{R}}, *, \diamond)$ *is a neutrosophic intuitionistic algebra fuzzy absolute value space.*

Proof. The range, definiteness, and symmetry conditions are immediate. For $x, y \geq 0$, the function $q(x) = x/(1+x)$ is increasing and satisfies

$$q(x) \diamond q(y) = q(x + y + xy).$$

Since

$$|\alpha + \beta| \leq |\alpha| + |\beta| \leq |\alpha| + |\beta| + |\alpha\beta|,$$

we obtain

$$q(\alpha + \beta) \leq q(\alpha) \diamond q(\beta).$$

Moreover, direct calculation gives

$$q(|\alpha\beta|) \leq q(|\alpha|) \diamond q(|\beta|),$$

because the difference of the corresponding cross-products equals $|\alpha| + |\beta| \geq 0$. Taking complements yields the required truth inequalities. Finally, for every $c \in [0,1]$ and $u, v \in [0,1]$,

$$c(u \diamond v) \leq (cu) \diamond (cv).$$

Applying this inequality with $c = \lambda$ and $c = 1 - \lambda$ gives the indeterminacy and falsity inequalities. □

Definition 3.5. *Let (p_n) be a sequence in a neutrosophic intuitionistic algebra fuzzy absolute value space $(\mathbb{R}, \Omega_{\mathbb{R}}, \Upsilon_{\mathbb{R}}, \Theta_{\mathbb{R}}, *, \diamond)$. The sequence (p_n) is said to* converge in the neutrosophic intuitionistic algebra fuzzy sense *to $p \in \mathbb{R}$ if, for every $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that*

$$\Omega_{\mathbb{R}}(p_n - p) > 1 - \varepsilon, \quad \Upsilon_{\mathbb{R}}(p_n - p) < \varepsilon, \quad \Theta_{\mathbb{R}}(p_n - p) < \varepsilon$$

for every $n \geq N$.

In this case, we write

$$p_n \longrightarrow p \quad or \quad \lim_{n\to\infty} p_n = p.$$

Equivalently,

$$\lim_{n\to\infty} \Omega_{\mathbb{R}}(p_n - p) = 1, \quad \lim_{n\to\infty} \Upsilon_{\mathbb{R}}(p_n - p) = 0, \quad \lim_{n\to\infty} \Theta_{\mathbb{R}}(p_n - p) = 0.$$

Definition 3.6. *A sequence (p_n) in a neutrosophic intuitionistic algebra fuzzy absolute value space is called a* neutrosophic intuitionistic fuzzy Cauchy sequence *if, for every $\varepsilon \in (0,1)$, there exists $N \in \mathbb{N}$ such that*

$$\Omega_{\mathbb{R}}(p_n - p_m) > 1 - \varepsilon, \quad \Upsilon_{\mathbb{R}}(p_n - p_m) < \varepsilon, \quad \Theta_{\mathbb{R}}(p_n - p_m) < \varepsilon$$

for every $m, n \geq N$.

Definition 3.7. *A neutrosophic intuitionistic algebra fuzzy absolute value space is called* complete with respect to the neutrosophic intuitionistic algebra fuzzy absolute value *if every neutrosophic intuitionistic fuzzy Cauchy sequence converges to an element of $\mathbb{R}$.*

3.3. **Neutrosophic Intuitionistic Algebra Fuzzy Normed Spaces.** We now extend the scalar structure to real algebras.

Definition 3.8. *Let U be a real algebra, let $*$ be a continuous t-norm, and let $\diamond$ be a continuous t-conorm. Suppose that $(\mathbb{R}, \Omega_{\mathbb{R}}, \Upsilon_{\mathbb{R}}, \Theta_{\mathbb{R}}, *, \diamond)$ is a neutrosophic intuitionistic algebra fuzzy absolute value space. Three mappings $\Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}} : \mathsf{U} \longrightarrow [0,1]$ are said to define a* neutrosophic intuitionistic algebra fuzzy norm *on U if, for all $\mathsf{L}, v \in \mathsf{U}$ and all $\alpha \in \mathbb{R}$, the following conditions hold:*

(1)
$$0 \leq \Omega_{\mathsf{U}}(\mathsf{L}), \Upsilon_{\mathsf{U}}(\mathsf{L}), \Theta_{\mathsf{U}}(\mathsf{L}) \leq 1;$$

(2) $\Omega_{\mathsf{U}}(\mathsf{L}) = 1 \Longleftrightarrow \mathsf{L} = 0$;
(3) $\Upsilon_{\mathsf{U}}(\mathsf{L}) = 0 \Longleftrightarrow \mathsf{L} = 0$;
(4) $\Theta_{\mathsf{U}}(\mathsf{L}) = 0 \Longleftrightarrow \mathsf{L} = 0$;

(5)
$$\Omega_{\mathsf{U}}(-\mathsf{L}) = \Omega_{\mathsf{U}}(\mathsf{L}), \quad \Upsilon_{\mathsf{U}}(-\mathsf{L}) = \Upsilon_{\mathsf{U}}(\mathsf{L}), \quad \Theta_{\mathsf{U}}(-\mathsf{L}) = \Theta_{\mathsf{U}}(\mathsf{L});$$

(6)
$$\Omega_{\mathsf{U}}(\alpha\mathsf{L}) \geq \Omega_{\mathbb{R}}(\alpha) * \Omega_{\mathsf{U}}(\mathsf{L});$$

(7)
$$\Upsilon_{\mathsf{U}}(\alpha\mathsf{L}) \leq \Upsilon_{\mathbb{R}}(\alpha) \diamond \Upsilon_{\mathsf{U}}(\mathsf{L});$$

(8)
$$\Theta_{\mathsf{U}}(\alpha\mathsf{L}) \leq \Theta_{\mathbb{R}}(\alpha) \diamond \Theta_{\mathsf{U}}(\mathsf{L});$$

(9)
$$\Omega_{\mathsf{U}}(\mathsf{L} + v) \geq \Omega_{\mathsf{U}}(\mathsf{L}) * \Omega_{\mathsf{U}}(v);$$

(10)
$$\Upsilon_{\mathsf{U}}(\mathsf{L} + v) \leq \Upsilon_{\mathsf{U}}(\mathsf{L}) \diamond \Upsilon_{\mathsf{U}}(v);$$

(11)
$$\Theta_{\mathsf{U}}(\mathsf{L} + v) \leq \Theta_{\mathsf{U}}(\mathsf{L}) \diamond \Theta_{\mathsf{U}}(v);$$

(12)
$$\Omega_{\mathsf{U}}(\mathsf{L}v) \geq \Omega_{\mathsf{U}}(\mathsf{L}) * \Omega_{\mathsf{U}}(v);$$

(13)
$$\Upsilon_{\mathsf{U}}(\mathsf{L}v) \leq \Upsilon_{\mathsf{U}}(\mathsf{L}) \diamond \Upsilon_{\mathsf{U}}(v);$$

(14)
$$\Theta_{\mathsf{U}}(\mathsf{L}v) \leq \Theta_{\mathsf{U}}(\mathsf{L}) \diamond \Theta_{\mathsf{U}}(v).$$

The system $(\mathsf{U}, \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$ *is called a* neutrosophic intuitionistic algebra fuzzy normed space*, abbreviated as an* NIAFNS*.*

Example 3.6. *Let* $\mathsf{U} = C[a, b]$*, equipped with pointwise addition, scalar multiplication, and pointwise multiplication. Suppose that* $(\mathbb{R}, \Omega_{\mathbb{R}}, \Upsilon_{\mathbb{R}}, \Theta_{\mathbb{R}}, *, \diamond)$ *is a neutrosophic intuitionistic algebra fuzzy absolute value space. For every* $f \in C[a, b]$*, define*

$$\Omega_{\mathsf{U}}(f) = \min_{x\in[a,b]} \Omega_{\mathbb{R}}(f(x)),$$

$$\Upsilon_{\mathsf{U}}(f) = \max_{x\in[a,b]} \Upsilon_{\mathbb{R}}(f(x)), \qquad \Theta_{\mathsf{U}}(f) = \max_{x\in[a,b]} \Theta_{\mathbb{R}}(f(x)).$$

Then, $(C[a, b], \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$ *is a neutrosophic intuitionistic algebra fuzzy normed space.*

Proof. The extrema exist because all scalar component functions are assumed continuous and $[a, b]$ is compact. The range, definiteness, and symmetry properties follow directly from the corresponding scalar properties. For example, for every $x \in [a, b]$,

$$\Omega_{\mathbb{R}}(f(x) + g(x)) \geq \Omega_{\mathbb{R}}(f(x)) * \Omega_{\mathbb{R}}(g(x)).$$

Taking the minimum over $x \in [a, b]$ and using monotonicity of $*$ gives

$$\Omega_{\mathsf{U}}(f + g) \geq \Omega_{\mathsf{U}}(f) * \Omega_{\mathsf{U}}(g).$$

Similarly,

$$\Upsilon_{\mathsf{U}}(f + g) \leq \Upsilon_{\mathsf{U}}(f) \diamond \Upsilon_{\mathsf{U}}(g),$$

and

$$\Theta_{\mathsf{U}}(f + g) \leq \Theta_{\mathsf{U}}(f) \diamond \Theta_{\mathsf{U}}(g).$$

The scalar multiplication and pointwise multiplication axioms are proved in exactly the same way. □

Lemma 3.2. *Let* $(\mathsf{U}, \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$ *be a neutrosophic intuitionistic algebra fuzzy normed space. Then, for all* $\mathsf{L}, v \in \mathsf{U}$*,*

$$\Omega_{\mathsf{U}}(\mathsf{L} - v) = \Omega_{\mathsf{U}}(v - \mathsf{L}),$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L} - v) = \Upsilon_{\mathsf{U}}(v - \mathsf{L}),$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - v) = \Theta_{\mathsf{U}}(v - \mathsf{L}).$$

Proof. Since $v - \mathsf{L} = -(\mathsf{L} - v)$, the assertions follow immediately from the symmetry axiom in Definition 3.8. □

3.4. **Convergence, Cauchy Sequences, and Completeness.** The notions of convergence, Cauchy sequences, and completeness are fundamental in the study of neutrosophic intuitionistic algebra fuzzy normed spaces. Convergence is described simultaneously through the three neutrosophic components: the truth-membership value of the difference tends to one, whereas its indeterminacy-membership and falsity-membership values tend to zero. The Cauchy condition expresses the same behaviour for pairs of sufficiently large terms without requiring the limit to be known in advance. Completeness ensures that every such Cauchy sequence converges to an element of the space.

Definition 3.9. *Let* (L_n) *be a sequence in a neutrosophic intuitionistic algebra fuzzy normed space* $(\mathsf{U}, \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$*. The sequence* (L_n) *is said to* converge in the neutrosophic intuitionistic algebra fuzzy sense *to* $\mathsf{L} \in \mathsf{U}$ *if, for every* $\varepsilon \in (0, 1)$*, there exists* $N \in \mathbb{N}$ *such that*

$$\Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) > 1 - \varepsilon,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) < \varepsilon, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) < \varepsilon$$

for every $n \geq N$*.*

In this case, we write

$$\mathsf{L}_n \longrightarrow \mathsf{L} \quad \textit{or} \quad \lim_{n\to\infty} \mathsf{L}_n = \mathsf{L}.$$

Equivalently,

$$\lim_{n\to\infty} \Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) = 1,$$

$$\lim_{n\to\infty} \Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) = 0, \qquad \lim_{n\to\infty} \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) = 0.$$

Definition 3.10. *A sequence* (L_n) *in a neutrosophic intuitionistic algebra fuzzy normed space is called a* neutrosophic intuitionistic fuzzy Cauchy sequence *if, for every* $\varepsilon \in (0, 1)$*, there exists* $N \in \mathbb{N}$ *such that*

$$\Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_m) > 1 - \varepsilon,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_m) < \varepsilon, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_m) < \varepsilon$$

for every $m, n \geq N$*.*

Definition 3.11. *A neutrosophic intuitionistic algebra fuzzy normed space is called* complete with respect to the neutrosophic intuitionistic algebra fuzzy norm *if every neutrosophic intuitionistic fuzzy Cauchy sequence converges to an element of the space.*

Theorem 3.1. *Every sequence in a neutrosophic intuitionistic algebra fuzzy normed space that converges in the neutrosophic intuitionistic algebra fuzzy sense is a neutrosophic intuitionistic fuzzy Cauchy sequence.*

Proof. Assume that $\mathsf{L}_n \to \mathsf{L}$, and let $\varepsilon \in (0,1)$. By continuity of $*$ and $\diamond$, there exists $\delta \in (0,1)$ such that

$$(1-\delta) * (1-\delta) > 1-\varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Since $\mathsf{L}_n \to \mathsf{L}$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) > 1-\delta,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) < \delta.$$

Let $m, n \geq N$. Since

$$\mathsf{L}_n - \mathsf{L}_m = (\mathsf{L}_n - \mathsf{L}) + (\mathsf{L} - \mathsf{L}_m),$$

Definition 3.8 and Lemma 3.2 give

$$\begin{aligned}
\Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_m) &\geq \Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) * \Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{L}_m) \\
&= \Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) * \Omega_{\mathsf{U}}(\mathsf{L}_m - \mathsf{L}) \\
&> (1-\delta) * (1-\delta) \\
&> 1-\varepsilon.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_m) &\leq \Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) \diamond \Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{L}_m) \\
&= \Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) \diamond \Upsilon_{\mathsf{U}}(\mathsf{L}_m - \mathsf{L}) \\
&< \delta \diamond \delta \\
&< \varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
\Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_m) &\leq \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) \diamond \Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{L}_m) \\
&= \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}) \diamond \Theta_{\mathsf{U}}(\mathsf{L}_m - \mathsf{L}) \\
&< \delta \diamond \delta \\
&< \varepsilon.
\end{aligned}$$

Therefore, (L_n) is a neutrosophic intuitionistic fuzzy Cauchy sequence. □

Definition 3.12. *A neutrosophic intuitionistic algebra fuzzy normed space is called* nondegenerate *if*

$$\Omega_{\mathsf{U}}(\mathsf{L}) > 0, \qquad \Upsilon_{\mathsf{U}}(\mathsf{L}) < 1, \qquad \Theta_{\mathsf{U}}(\mathsf{L}) < 1$$

for every $\mathsf{L} \in \mathsf{U}$*, and if the operations satisfy*

$$a * b > 0 \quad \textit{whenever} \quad a, b > 0,$$

and

$$a \diamond b < 1 \quad \textit{whenever} \quad a, b < 1.$$

Remark 3.4. *The product t-norm and algebraic-sum t-conorm satisfy the two regularity conditions in Definition 3.12. The same is true for the minimum t-norm and maximum t-conorm.*

Definition 3.13. *A sequence* (L_n) *in a neutrosophic intuitionistic algebra fuzzy normed space is called* bounded in the neutrosophic intuitionistic algebra fuzzy sense *if there exists* $r \in (0,1)$ *such that*

$$\Omega_{\mathsf{U}}(\mathsf{L}_n) > 1-r,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n) < r, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n) < r$$

for every $n \in \mathbb{N}$.

Theorem 3.2. *Every neutrosophic intuitionistic fuzzy Cauchy sequence in a nondegenerate neutrosophic intuitionistic algebra fuzzy normed space is bounded in the neutrosophic intuitionistic algebra fuzzy sense.*

Proof. Let (L_n) be a neutrosophic intuitionistic fuzzy Cauchy sequence. Taking $\varepsilon = 1/2$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$,

$$\Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_N) > \frac{1}{2},$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_N) < \frac{1}{2}, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_N) < \frac{1}{2}.$$

Set

$$a = \Omega_{\mathsf{U}}(\mathsf{L}_N) * \frac{1}{2},$$

$$b = \Upsilon_{\mathsf{U}}(\mathsf{L}_N) \diamond \frac{1}{2}, \qquad c = \Theta_{\mathsf{U}}(\mathsf{L}_N) \diamond \frac{1}{2}.$$

By nondegeneracy, $a > 0$ and $b, c < 1$. For every $n \geq N$,

$$\begin{aligned} \Omega_{\mathsf{U}}(\mathsf{L}_n) &= \Omega_{\mathsf{U}}\big(\mathsf{L}_N + (\mathsf{L}_n - \mathsf{L}_N)\big) \\ &\geq \Omega_{\mathsf{U}}(\mathsf{L}_N) * \Omega_{\mathsf{U}}(\mathsf{L}_n - \mathsf{L}_N) \\ &\geq a. \end{aligned}$$

Similarly,

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n) \leq b, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n) \leq c.$$

Choose $r_1 \in (0,1)$ such that

$$\max\{1-a, b, c\} < r_1 < 1.$$

Then, for every $n \geq N$,

$$\Omega_{\mathsf{U}}(\mathsf{L}_n) > 1 - r_1,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_n) < r_1, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n) < r_1.$$

For the finite set $\{\mathsf{L}_1, \ldots, \mathsf{L}_{N-1}\}$, nondegeneracy ensures that there exists $r_2 \in (0,1)$ such that

$$\Omega_{\mathsf{U}}(\mathsf{L}_k) > 1 - r_2,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_k) < r_2, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_k) < r_2$$

for every $1 \leq k < N$. Let

$$r = \max\{r_1, r_2\}.$$

Then $r \in (0,1)$ and the required three inequalities hold for every $n \in \mathbb{N}$. Hence, (L_n) is bounded in the neutrosophic intuitionistic algebra fuzzy sense. □

3.5. **Neighbourhoods, Balls, and Topological Concepts.** The neutrosophic intuitionistic algebra fuzzy norm naturally induces a topological structure on the underlying algebra. The degree of closeness between two points is determined simultaneously by the truth-membership, indeterminacy-membership, and falsity-membership values of their difference. This leads to the notions of neutrosophic intuitionistic fuzzy neighbourhoods, open and closed balls, interior points, adherent points, open and closed subsets, closure, and density.

Definition 3.14. *Let* $(\mathsf{U}, \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$ *be a neutrosophic intuitionistic algebra fuzzy normed space. For* $\mathsf{L} \in \mathsf{U}$ *and* $r \in (0,1)$*, the set*

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) = \left\{ v \in \mathsf{U} : \begin{array}{l} \Omega_{\mathsf{U}}(\mathsf{L} - v) > 1 - r, \\ \Upsilon_{\mathsf{U}}(\mathsf{L} - v) < r, \\ \Theta_{\mathsf{U}}(\mathsf{L} - v) < r \end{array} \right\}$$

is called the neutrosophic intuitionistic fuzzy open ball *or* neutrosophic intuitionistic fuzzy neighbourhood *of* L *with radius* r*.*

Definition 3.15. *The neutrosophic intuitionistic fuzzy closed ball with centre* $\mathsf{L} \in \mathsf{U}$ *and radius* $r \in (0,1)$ *is defined by*

$$\overline{B}_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) = \left\{ v \in \mathsf{U} : \begin{array}{l} \Omega_{\mathsf{U}}(\mathsf{L} - v) \geq 1 - r, \\ \Upsilon_{\mathsf{U}}(\mathsf{L} - v) \leq r, \\ \Theta_{\mathsf{U}}(\mathsf{L} - v) \leq r \end{array} \right\}.$$

Remark 3.5. *The inequalities*

$$\Omega_{\mathsf{U}}(\mathsf{L} - v) > 1 - r, \quad \Upsilon_{\mathsf{U}}(\mathsf{L} - v) < r, \quad \Theta_{\mathsf{U}}(\mathsf{L} - v) < r$$

mean that v *is sufficiently close to* L *in the neutrosophic intuitionistic algebra fuzzy sense. Thus, these neighbourhoods play the same role as ordinary open balls in classical normed spaces.*

Definition 3.16. *Let* $D \subseteq \mathsf{U}$*. A point* $\mathsf{L} \in D$ *is called a* neutrosophic intuitionistic fuzzy interior point *of* D *if there exists* $r \in (0,1)$ *such that*

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) \subseteq D.$$

The set of all neutrosophic intuitionistic fuzzy interior points of D *is called the* neutrosophic intuitionistic fuzzy interior *of* D *and is denoted by* $\mathrm{Int}(D)$*.*

Definition 3.17. *Let* $D \subseteq \mathsf{U}$*. A point* $\mathsf{L} \in \mathsf{U}$ *is called a* neutrosophic intuitionistic fuzzy adherent point *of* D *if, for every* $r \in (0,1)$*,*

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) \cap D \neq \varnothing.$$

The set of all neutrosophic intuitionistic fuzzy adherent points of D *is called the* neutrosophic intuitionistic fuzzy closure *of* D *and is denoted by* $\overline{D}$*.*

Definition 3.18. *A subset* $W \subseteq \mathsf{U}$ *is called* open in the neutrosophic intuitionistic fuzzy topology *if, for every* $w \in W$*, there exists* $r \in (0,1)$ *such that*

$$B_{\Omega,\Upsilon,\Theta}(w, r) \subseteq W.$$

A subset $D \subseteq \mathsf{U}$ *is called* closed in the neutrosophic intuitionistic fuzzy topology *if its complement* $D^c = \mathsf{U} \setminus D$ *is open in the neutrosophic intuitionistic fuzzy topology.*

Definition 3.19. *A subset* $D \subseteq \mathsf{U}$ *is called* dense in the neutrosophic intuitionistic fuzzy topology of U *if*

$$\overline{D} = \mathsf{U}.$$

3.6. **Fundamental Topological Properties.** We now prove that the neutrosophic intuitionistic fuzzy open balls generate a natural topology on the space.

Theorem 3.3. *Every neutrosophic intuitionistic fuzzy open ball* $B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r)$ *is an open subset of the neutrosophic intuitionistic fuzzy topology on* U*.*

Proof. Let $\mathsf{P} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r)$. Set

$$a = \Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{P}), \quad b = \Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{P}), \quad c = \Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{P}).$$

Then

$$a > 1 - r, \qquad b < r, \qquad c < r.$$

Since $a * 1 = a > 1 - r$ and $*$ is continuous, there exists $\delta_1 \in (0,1)$ such that

$$a * (1 - \delta_1) > 1 - r.$$

Since $b \diamond 0 = b < r$ and $c \diamond 0 = c < r$, continuity of $\diamond$ gives $\delta_2, \delta_3 \in (0,1)$ such that

$$b \diamond \delta_2 < r, \qquad c \diamond \delta_3 < r.$$

Let

$$\delta = \min\{\delta_1, \delta_2, \delta_3\}.$$

We show that

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{P},\delta) \subseteq B_{\Omega,\Upsilon,\Theta}(\mathsf{L},r).$$

Take $\mathsf{Q} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{P},\delta)$. Then

$$\Omega_{\mathsf{U}}(\mathsf{P}-\mathsf{Q}) > 1-\delta,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{P}-\mathsf{Q}) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{P}-\mathsf{Q}) < \delta.$$

As

$$\mathsf{L}-\mathsf{Q} = (\mathsf{L}-\mathsf{P})+(\mathsf{P}-\mathsf{Q}),$$

we obtain

$$\begin{aligned}\Omega_{\mathsf{U}}(\mathsf{L}-\mathsf{Q}) &\geq \Omega_{\mathsf{U}}(\mathsf{L}-\mathsf{P}) * \Omega_{\mathsf{U}}(\mathsf{P}-\mathsf{Q})\\ &\geq a*(1-\delta)\\ &\geq a*(1-\delta_1)\\ &> 1-r.\end{aligned}$$

Similarly,

$$\begin{aligned}\Upsilon_{\mathsf{U}}(\mathsf{L}-\mathsf{Q}) &\leq \Upsilon_{\mathsf{U}}(\mathsf{L}-\mathsf{P}) \diamond \Upsilon_{\mathsf{U}}(\mathsf{P}-\mathsf{Q})\\ &\leq b\diamond\delta\\ &\leq b\diamond\delta_2\\ &< r,\end{aligned}$$

and

$$\begin{aligned}\Theta_{\mathsf{U}}(\mathsf{L}-\mathsf{Q}) &\leq \Theta_{\mathsf{U}}(\mathsf{L}-\mathsf{P}) \diamond \Theta_{\mathsf{U}}(\mathsf{P}-\mathsf{Q})\\ &\leq c\diamond\delta\\ &\leq c\diamond\delta_3\\ &< r.\end{aligned}$$

Therefore, $\mathsf{Q} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{L},r)$. Hence every point of the ball has a neutrosophic intuitionistic fuzzy neighbourhood contained in the ball, and the ball is open. □

Theorem 3.4. *Let $(\mathsf{U},\Omega_{\mathsf{U}},\Upsilon_{\mathsf{U}},\Theta_{\mathsf{U}},*,\diamond)$ be a neutrosophic intuitionistic algebra fuzzy normed space, and let $D \subseteq \mathsf{U}$. Then $d \in \overline{D}$ if and only if there exists a sequence (d_n) in D such that $d_n \to d$ in the neutrosophic intuitionistic algebra fuzzy norm.*

Proof. Suppose first that $d \in \overline{D}$. Then, for every $n \in \mathbb{N}$,

$$B_{\Omega,\Upsilon,\Theta}\left(d,\frac{1}{n+1}\right) \cap D \neq \varnothing.$$

Choose

$$d_n \in B_{\Omega,\Upsilon,\Theta}\left(d,\frac{1}{n+1}\right) \cap D.$$

Then

$$\Omega_{\mathsf{U}}(d-d_n) > 1-\frac{1}{n+1},$$
$$\Upsilon_{\mathsf{U}}(d-d_n) < \frac{1}{n+1}, \qquad \Theta_{\mathsf{U}}(d-d_n) < \frac{1}{n+1}.$$

By Lemma 3.2, the same inequalities hold with $d_n - d$ in place of $d - d_n$. Let $\varepsilon \in (0,1)$ and choose $N \in \mathbb{N}$ such that $1/(N+1) < \varepsilon$. For every $n \geq N$,

$$\Omega_{\mathsf{U}}(d_n-d) > 1-\frac{1}{n+1} \geq 1-\frac{1}{N+1} > 1-\varepsilon,$$

and

$$\Upsilon_{\mathsf{U}}(d_n-d) < \frac{1}{n+1} \leq \frac{1}{N+1} < \varepsilon,$$

$$\Theta_{\mathsf{U}}(d_n - d) < \frac{1}{n+1} \leq \frac{1}{N+1} < \varepsilon.$$

Therefore, $d_n \to d$.

Conversely, suppose that there exists a sequence (d_n) in D such that $d_n \to d$. Let $r \in (0,1)$. By convergence, there exists $N \in \mathbb{N}$ such that

$$\Omega_{\mathsf{U}}(d_n - d) > 1 - r,$$
$$\Upsilon_{\mathsf{U}}(d_n - d) < r, \qquad \Theta_{\mathsf{U}}(d_n - d) < r$$

for every $n \geq N$. By Lemma 3.2,

$$\Omega_{\mathsf{U}}(d - d_n) > 1 - r,$$
$$\Upsilon_{\mathsf{U}}(d - d_n) < r, \qquad \Theta_{\mathsf{U}}(d - d_n) < r.$$

Thus,

$$d_n \in B_{\Omega,\Upsilon,\Theta}(d, r) \cap D$$

for every $n \geq N$. Hence $B_{\Omega,\Upsilon,\Theta}(d, r) \cap D \neq \varnothing$. Since $r \in (0,1)$ was arbitrary, $d \in \overline{D}$. □

Corollary 3.1. *A subset D of a neutrosophic intuitionistic algebra fuzzy normed space is closed in the neutrosophic intuitionistic fuzzy topology if and only if, whenever a sequence (d_n) in D converges to $d \in \mathsf{U}$, one has $d \in D$.*

Proof. The result follows directly from Theorem 3.4 and the identity $D = \overline{D}$ for closed subsets. □

Corollary 3.2. *A subset $D \subseteq \mathsf{U}$ is dense in the neutrosophic intuitionistic fuzzy topology of U if and only if, for every $u \in \mathsf{U}$, there exists a sequence (d_n) in D such that $d_n \to u$.*

Proof. By Definition 3.19,

$$D \text{ is dense} \quad \Longleftrightarrow \quad \overline{D} = \mathsf{U}.$$

The result now follows from Theorem 3.4. □

4. Compactness in Neutrosophic Intuitionistic Algebra Fuzzy Normed Spaces

In this section, we introduce compactness-related concepts in neutrosophic intuitionistic algebra fuzzy normed spaces. We begin with neutrosophic intuitionistic fuzzy open covers and then formulate compactness, boundedness, and total boundedness in terms of the topology generated by the three-component open balls. Since closeness is described simultaneously by truth-membership, indeterminacy-membership, and falsity-membership values, every covering and sequential condition must incorporate all three components. We subsequently establish the relationships among total boundedness, boundedness, compactness, and completeness.

Throughout this section, $(\mathsf{U}, \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$ denotes a neutrosophic intuitionistic algebra fuzzy normed space. The open ball with centre $\mathsf{p} \in \mathsf{U}$ and radius $r \in (0,1)$ is written as

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r) = \left\{ \mathsf{L} \in \mathsf{U} : \begin{array}{c} \Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{p}) > 1 - r, \\ \Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{p}) < r, \\ \Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{p}) < r \end{array} \right\}.$$

4.1. Open Covers and Compactness.

Definition 4.1. *Let $P \subseteq \mathsf{U}$ and let $\mathcal{O} = \{O_j : j \in J\}$, where every O_j is a neutrosophic intuitionistic fuzzy open subset of U. The family $\mathcal{O}$ is called a* neutrosophic intuitionistic fuzzy open cover *of P if*

$$P \subseteq \bigcup_{j \in J} O_j.$$

Equivalently, for every $\mathsf{p} \in P$, there exists $j \in J$ such that $\mathsf{p} \in O_j$. A finite subcollection of $\mathcal{O}$ that still covers P is called a finite neutrosophic intuitionistic fuzzy open subcover *of P.*

Definition 4.2. *The space* U *is called* compact in the neutrosophic intuitionistic algebra fuzzy topology*, or briefly* NIAF-compact*, if every neutrosophic intuitionistic fuzzy open cover of* U *admits a finite subcover. Thus, whenever*

$$\mathcal{O} = \{O_j : j \in J\}$$

is a neutrosophic intuitionistic fuzzy open cover of U*, there exist* $j_1, j_2, \ldots, j_k \in J$ *such that*

$$\mathsf{U} = \bigcup_{i=1}^{k} O_{j_i}.$$

Definition 4.3. *A subset* $P \subseteq \mathsf{U}$ *is called* NIAF-compact *if every neutrosophic intuitionistic fuzzy open cover of* P *has a finite subcover.*

Example 4.1. *Let* $\lambda \in (0,1)$ *and define, for every* $x \in \mathbb{R}$, $q(x) = \frac{|x|}{1+|x|}$, *and*

$$\Omega_{\mathbb{R}}(x) = \frac{1}{1+|x|}, \qquad \Upsilon_{\mathbb{R}}(x) = \lambda q(x), \qquad \Theta_{\mathbb{R}}(x) = (1-\lambda)q(x).$$

Take the product t-norm and the algebraic-sum t-conorm. By Example 3.5, these functions determine a neutrosophic intuitionistic algebra fuzzy absolute value on $\mathbb{R}$.

For $x, y \in \mathbb{R}$ *and* $r \in (0,1)$*, we have*

$$\Omega_{\mathbb{R}}(x-y) > 1-r \quad \Longleftrightarrow \quad q(x-y) < r \quad \Longleftrightarrow \quad |x-y| < \frac{r}{1-r}.$$

Whenever $q(x-y) < r$*, both* $\Upsilon_{\mathbb{R}}(x-y) < r$ *and* $\Theta_{\mathbb{R}}(x-y) < r$ *also hold. Consequently, the NIAF topology on* $\mathbb{R}$ *coincides with its usual topology.*

Consider

$$P = (0,1).$$

The family

$$\mathcal{O} = \left\{ \left(\frac{1}{k}, 1 \right) : k = 2, 3, \ldots \right\}$$

is a neutrosophic intuitionistic fuzzy open cover of $(0,1)$*. However, every finite subcollection has a smallest index endpoint strictly greater than zero and therefore fails to cover points of* $(0,1)$ *sufficiently close to zero. Hence,* $(0,1)$ *is not NIAF-compact.*

Remark 4.1. *Every finite subset of a neutrosophic intuitionistic algebra fuzzy normed space is NIAF-compact.*

4.2. **Boundedness and Total Boundedness.**

Definition 4.4. *A nonempty subset* $P \subseteq \mathsf{U}$ *is called* bounded in the neutrosophic intuitionistic algebra fuzzy sense*, or briefly* NIAF-bounded*, if there exists* $r \in (0,1)$ *such that, for all* $\mathsf{L}, v \in P$*,*

$$\Omega_{\mathsf{U}}(\mathsf{L} - v) > 1 - r, \ \Upsilon_{\mathsf{U}}(\mathsf{L} - v) < r, \qquad \Theta_{\mathsf{U}}(\mathsf{L} - v) < r.$$

Definition 4.5. *A nonempty subset* $P \subseteq \mathsf{U}$ *is called* totally bounded in the neutrosophic intuitionistic algebra fuzzy sense*, or briefly* NIAF-totally bounded*, if, for every* $r \in (0,1)$*, there exists a finite subset*

$$A_r = \{\mathsf{p}_1, \mathsf{p}_2, \ldots, \mathsf{p}_k\} \subseteq P$$

such that

$$P \subseteq \bigcup_{j=1}^{k} B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_j, r).$$

Equivalently, for every $\mathsf{L} \in P$*, there exists* $j \in \{1, 2, \ldots, k\}$ *satisfying*

$$\Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_j) > 1 - r,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}-\mathsf{p}_j)<r, \qquad \Theta_{\mathsf{U}}(\mathsf{L}-\mathsf{p}_j)<r.$$

The finite set A_r is called a finite NIAF r-net *for P.*

Remark 4.2. *The present algebra fuzzy norm is defined by three mappings $\Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}} : \mathsf{U} \longrightarrow [0,1]$. Therefore, no additional time parameter is required in Definition 4.5. The three inequalities in the definition state that each point of P is close to one of finitely many centres: its truth-membership degree is sufficiently high, whereas its indeterminacy- and falsity-membership degrees are sufficiently low.*

Example 4.2. *Consider the neutrosophic intuitionistic algebra fuzzy absolute value on $\mathbb{R}$ from Example 3.5, namely*

$$\Omega_{\mathbb{R}}(x)=\frac{1}{1+|x|}, \qquad \Upsilon_{\mathbb{R}}(x)=\lambda\frac{|x|}{1+|x|}, \qquad \Theta_{\mathbb{R}}(x)=(1-\lambda)\frac{|x|}{1+|x|},$$

where $\lambda \in (0,1)$. We show that

$$P=[0,1]$$

is NIAF-totally bounded.

Let $r \in (0,1)$ and set

$$\delta=\frac{r}{1-r}.$$

If $|x-y|<\delta$, then

$$\frac{|x-y|}{1+|x-y|}<r.$$

Consequently,

$$\Omega_{\mathbb{R}}(x-y)>1-r,$$
$$\Upsilon_{\mathbb{R}}(x-y)<r, \qquad \Theta_{\mathbb{R}}(x-y)<r.$$

Choose $N \in \mathbb{N}$ such that

$$\frac{1}{N}<\delta,$$

and set

$$\mathsf{p}_i=\frac{i}{N}, \qquad i=0,1,\dots,N.$$

For every $x \in [0,1]$, there exists $i \in \{0,1,\dots,N\}$ such that

$$|x-\mathsf{p}_i|<\frac{1}{N}<\delta.$$

Hence,

$$x \in B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_i,r).$$

It follows that

$$[0,1] \subseteq \bigcup_{i=0}^{N} B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_i,r).$$

Thus, $[0,1]$ is NIAF-totally bounded.

The systematic study of fuzzy normed linear spaces was initiated by Felbin [5]. Bag and Samanta subsequently investigated boundedness and bounded linear operators in fuzzy normed linear spaces [2]. Saadati and Park developed intuitionistic fuzzy topological structures using continuous t-norms and t-conorms [15], while Sadeqi and Kia studied the topological structure of fuzzy normed linear spaces [16]. Compactness in fuzzy normed spaces was also considered by Deschrijver, O'Regan, Saadati, and Vaezpour [4]. Motivated by these works, we establish the corresponding relationships in the present three-component setting.

Proposition 4.1. *Every NIAF-totally bounded subset of a nondegenerate neutrosophic intuitionistic algebra fuzzy normed space is NIAF-bounded.*

Proof. Let $P \subseteq \mathsf{U}$ be NIAF-totally bounded. Fix $\zeta \in (0,1)$. There exists a finite NIAF ζ-net

$$A = \{\mathsf{p}_1, \mathsf{p}_2, \ldots, \mathsf{p}_k\} \subseteq P.$$

Since A is finite, define

$$a = \min_{1 \le i,j \le k} \Omega_{\mathsf{U}}(\mathsf{p}_i - \mathsf{p}_j),$$

$$b = \max_{1 \le i,j \le k} \Upsilon_{\mathsf{U}}(\mathsf{p}_i - \mathsf{p}_j), \qquad c = \max_{1 \le i,j \le k} \Theta_{\mathsf{U}}(\mathsf{p}_i - \mathsf{p}_j).$$

By nondegeneracy,

$$a > 0, \qquad b < 1, \qquad c < 1.$$

Let $\mathsf{L}, v \in P$. Choose $\mathsf{p}_i, \mathsf{p}_j \in A$ such that

$$\Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_i) > 1 - \zeta, \qquad \Omega_{\mathsf{U}}(v - \mathsf{p}_j) > 1 - \zeta,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_i) < \zeta, \qquad \Upsilon_{\mathsf{U}}(v - \mathsf{p}_j) < \zeta,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_i) < \zeta, \qquad \Theta_{\mathsf{U}}(v - \mathsf{p}_j) < \zeta.$$

Since

$$\mathsf{L} - v = (\mathsf{L} - \mathsf{p}_i) + (\mathsf{p}_i - \mathsf{p}_j) + (\mathsf{p}_j - v),$$

Definition 3.8 and Lemma 3.2 yield

$$\begin{aligned} \Omega_{\mathsf{U}}(\mathsf{L} - v) &\ge \Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_i) * \Omega_{\mathsf{U}}(\mathsf{p}_i - \mathsf{p}_j) * \Omega_{\mathsf{U}}(\mathsf{p}_j - v) \\ &\ge (1 - \zeta) * a * (1 - \zeta). \end{aligned}$$

Similarly,

$$\Upsilon_{\mathsf{U}}(\mathsf{L} - v) \le \zeta \diamond b \diamond \zeta,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - v) \le \zeta \diamond c \diamond \zeta.$$

Set

$$A_0 = (1 - \zeta) * a * (1 - \zeta),$$

$$B_0 = \zeta \diamond b \diamond \zeta, \qquad C_0 = \zeta \diamond c \diamond \zeta.$$

By the regularity conditions in Definition 3.12, we have

$$A_0 > 0, \qquad B_0 < 1, \qquad C_0 < 1.$$

Choose $r \in (0,1)$ such that

$$\max\{1 - A_0, B_0, C_0\} < r < 1.$$

Then, for all $\mathsf{L}, v \in P$,

$$\Omega_{\mathsf{U}}(\mathsf{L} - v) > 1 - r,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L} - v) < r, \qquad \Theta_{\mathsf{U}}(\mathsf{L} - v) < r.$$

Therefore, P is NIAF-bounded. □

4.3. **Sequential Characterization of Total Boundedness.**

Theorem 4.1. *Let $P \subseteq \mathsf{U}$. Then P is NIAF-totally bounded if and only if every sequence in P has a neutrosophic intuitionistic fuzzy Cauchy subsequence.*

Proof. Assume first that P is NIAF-totally bounded, and let (p_k) be a sequence in P. For every $m \in \mathbb{N}$, choose $\delta_m \in (0,1)$ such that

$$(1-\delta_m) * (1-\delta_m) > 1 - \frac{1}{m+1} \quad \text{and} \quad \delta_m \diamond \delta_m < \frac{1}{m+1}.$$

Such a choice is possible by the continuity of $*$ and $\diamond$.

Since P is NIAF-totally bounded, it has a finite δ_1-net. At least one ball determined by this net contains infinitely many terms of (p_k). Let $(\mathsf{p}_k^{(1)})$ be a subsequence consisting of those terms. Applying total boundedness again, one of the balls from a finite δ_2-net contains infinitely many terms of $(\mathsf{p}_k^{(1)})$. Denote the resulting subsequence by $(\mathsf{p}_k^{(2)})$. Continuing inductively, we obtain nested subsequences

$$(\mathsf{p}_k) \supseteq (\mathsf{p}_k^{(1)}) \supseteq (\mathsf{p}_k^{(2)}) \supseteq \cdots$$

such that all terms of $(\mathsf{p}_k^{(m)})$ lie in a ball

$$B_{\Omega,\Upsilon,\Theta}(a_m, \delta_m)$$

for some $a_m \in P$.

Choose the diagonal subsequence (q_m) so that q_m is a term of $(\mathsf{p}_k^{(m)})$ and its index in the original sequence is strictly larger than the index chosen at the preceding step. Then, for every fixed N, all q_m with $m \geq N$ belong to $(\mathsf{p}_k^{(N)})$.

Let $\varepsilon \in (0,1)$ and choose $N \in \mathbb{N}$ such that

$$\frac{1}{N+1} < \varepsilon.$$

For $m, n \geq N$, both q_m and q_n belong to $B_{\Omega,\Upsilon,\Theta}(a_N, \delta_N)$. Hence,

$$\Omega_{\mathsf{U}}(q_m - a_N) > 1 - \delta_N, \qquad \Omega_{\mathsf{U}}(q_n - a_N) > 1 - \delta_N,$$
$$\Upsilon_{\mathsf{U}}(q_m - a_N) < \delta_N, \qquad \Upsilon_{\mathsf{U}}(q_n - a_N) < \delta_N,$$

and

$$\Theta_{\mathsf{U}}(q_m - a_N) < \delta_N, \qquad \Theta_{\mathsf{U}}(q_n - a_N) < \delta_N.$$

Using

$$q_m - q_n = (q_m - a_N) + (a_N - q_n)$$

and Lemma 3.2, we obtain

$$\begin{aligned} \Omega_{\mathsf{U}}(q_m - q_n) &\geq \Omega_{\mathsf{U}}(q_m - a_N) * \Omega_{\mathsf{U}}(a_N - q_n) \\ &> (1-\delta_N) * (1-\delta_N) \\ &> 1 - \frac{1}{N+1} \\ &> 1 - \varepsilon. \end{aligned}$$

Similarly,

$$\Upsilon_{\mathsf{U}}(q_m - q_n) < \delta_N \diamond \delta_N < \frac{1}{N+1} < \varepsilon,$$

and

$$\Theta_{\mathsf{U}}(q_m - q_n) < \delta_N \diamond \delta_N < \frac{1}{N+1} < \varepsilon.$$

Thus, (q_m) is a neutrosophic intuitionistic fuzzy Cauchy subsequence.

Conversely, assume that every sequence in P has a neutrosophic intuitionistic fuzzy Cauchy subsequence. Suppose, contrary to the claim, that P is not NIAF-totally bounded. Then there exists $\zeta \in (0,1)$ for which P has no finite NIAF ζ-net.

Choose $\mathsf{p}_1 \in P$. Since $\{\mathsf{p}_1\}$ is not a finite NIAF ζ-net, there exists

$$\mathsf{p}_2 \in P \setminus B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_1, \zeta).$$

Inductively, after choosing $\mathsf{p}_1, \ldots, \mathsf{p}_n$, select

$$\mathsf{p}_{n+1} \in P \setminus \bigcup_{j=1}^{n} B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_j, \zeta).$$

Thus, whenever $m > j$,

$$\mathsf{p}_m \notin B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_j, \zeta).$$

Consequently, at least one of the following inequalities holds:

$$\Omega_{\mathsf{U}}(\mathsf{p}_m - \mathsf{p}_j) \leq 1 - \zeta,$$

$$\Upsilon_{\mathsf{U}}(\mathsf{p}_m - \mathsf{p}_j) \geq \zeta, \quad \text{or} \quad \Theta_{\mathsf{U}}(\mathsf{p}_m - \mathsf{p}_j) \geq \zeta.$$

Therefore, no subsequence of (p_k) satisfies the Cauchy condition for $\varepsilon = \zeta$, which contradicts the assumption. Hence, P is NIAF-totally bounded. □

4.4. **Relations Among Compactness, Total Boundedness, and Completeness.**

Proposition 4.2. *Every NIAF-compact neutrosophic intuitionistic algebra fuzzy normed space is NIAF-totally bounded.*

Proof. Let $r \in (0,1)$. The family $\mathcal{B}_r = \{B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r) : \mathsf{p} \in \mathsf{U}\}$ is a neutrosophic intuitionistic fuzzy open cover of U. By NIAF-compactness, there exist $\mathsf{p}_1, \ldots, \mathsf{p}_k \in \mathsf{U}$ such that

$$\mathsf{U} = \bigcup_{j=1}^{k} B_{\Omega,\Upsilon,\Theta}(\mathsf{p}_j, r).$$

Therefore, $\{\mathsf{p}_1, \ldots, \mathsf{p}_k\}$ is a finite NIAF r-net for U. Since r was arbitrary, U is NIAF-totally bounded. □

Theorem 4.2. *Every NIAF-compact subset P of U is sequentially compact; that is, every sequence in P has a subsequence converging to a point of P in the neutrosophic intuitionistic algebra fuzzy sense.*

Proof. Let (p_n) be a sequence in P. For each $m \in \mathbb{N}$, set $A_m = \{\mathsf{p}_n : n \geq m\}$ and let F_m be the closure of A_m relative to P. Each F_m is a nonempty closed subset of the compact space P, and

$$F_1 \supseteq F_2 \supseteq F_3 \supseteq \cdots.$$

Hence, the family (F_m) has the finite-intersection property. Compactness gives

$$\bigcap_{m=1}^{\infty} F_m \neq \varnothing.$$

Choose

$$\mathsf{p} \in \bigcap_{m=1}^{\infty} F_m.$$

We construct a subsequence recursively. Choose $\mathsf{p}_{n_1} \in A_1 \cap B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, 1/2)$. Suppose that $n_1 < n_2 < \cdots < n_{k-1}$ have been selected. Since $\mathsf{p} \in F_{n_{k-1}+1}$, the ball $B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, 1/(k+1))$ intersects $A_{n_{k-1}+1}$. Select

$$\mathsf{p}_{n_k} \in A_{n_{k-1}+1} \cap B_{\Omega,\Upsilon,\Theta}\left(\mathsf{p}, \frac{1}{k+1}\right).$$

Then, $n_k > n_{k-1}$ and

$$\Omega_{\mathsf{U}}(\mathsf{p}_{n_k} - \mathsf{p}) > 1 - \frac{1}{k+1},$$
$$\Upsilon_{\mathsf{U}}(\mathsf{p}_{n_k} - \mathsf{p}) < \frac{1}{k+1}, \qquad \Theta_{\mathsf{U}}(\mathsf{p}_{n_k} - \mathsf{p}) < \frac{1}{k+1}.$$

Therefore,

$$\mathsf{p}_{n_k} \longrightarrow \mathsf{p}.$$

□

Proposition 4.3. *Every NIAF-compact neutrosophic intuitionistic algebra fuzzy normed space is complete with respect to its neutrosophic intuitionistic algebra fuzzy norm.*

Proof. Let (p_k) be a neutrosophic intuitionistic fuzzy Cauchy sequence in U. By Theorem 4.2, there exist a subsequence (p_{k_j}) and a point $\mathsf{p} \in \mathsf{U}$ such that

$$\mathsf{p}_{k_j} \longrightarrow \mathsf{p}.$$

Let $\varepsilon \in (0,1)$. Choose $\delta \in (0,1)$ such that

$$(1-\delta) * (1-\delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

Since (p_k) is Cauchy, there exists $N_1 \in \mathbb{N}$ such that, for all $k, m \geq N_1$,

$$\Omega_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_m) > 1 - \delta,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_m) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_m) < \delta.$$

Since $\mathsf{p}_{k_j} \to \mathsf{p}$, choose j such that $k_j \geq N_1$ and

$$\Omega_{\mathsf{U}}(\mathsf{p}_{k_j} - \mathsf{p}) > 1 - \delta,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{p}_{k_j} - \mathsf{p}) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{p}_{k_j} - \mathsf{p}) < \delta.$$

For every $k \geq N_1$,

$$\mathsf{p}_k - \mathsf{p} = (\mathsf{p}_k - \mathsf{p}_{k_j}) + (\mathsf{p}_{k_j} - \mathsf{p}).$$

Therefore,

$$\begin{aligned}\Omega_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}) &\geq \Omega_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_{k_j}) * \Omega_{\mathsf{U}}(\mathsf{p}_{k_j} - \mathsf{p}) \\ &> (1-\delta) * (1-\delta) \\ &> 1 - \varepsilon.\end{aligned}$$

Similarly,

$$\Upsilon_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}) < \delta \diamond \delta < \varepsilon,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}) < \delta \diamond \delta < \varepsilon.$$

Hence, $\mathsf{p}_k \to \mathsf{p}$. Thus, U is complete. □

Theorem 4.3. *If U is NIAF-totally bounded and complete with respect to its neutrosophic intuitionistic algebra fuzzy norm, then U is NIAF-compact.*

Proof. Let $\mathcal{O} = \{O_\lambda : \lambda \in \Lambda\}$ be a neutrosophic intuitionistic fuzzy open cover of U. Assume, contrary to the assertion, that $\mathcal{O}$ has no finite subcover.

For each $n \in \mathbb{N}$, choose a finite NIAF $1/(n+1)$-net

$$A_n = \{\mathsf{p}_{n,1}, \ldots, \mathsf{p}_{n,k_n}\}$$

for U. Thus,

$$\mathsf{U} = \bigcup_{i=1}^{k_n} B_{\Omega,\Upsilon,\Theta}\left(\mathsf{p}_{n,i}, \frac{1}{n+1}\right).$$

For every n, at least one of these finitely many balls is not contained in any single member of $\mathcal{O}$. Indeed, otherwise one could choose, for each ball, a member of $\mathcal{O}$ containing it, and the resulting finite collection would cover U. Choose one such ball and denote its centre by p_n. Hence,

$$B_{\Omega,\Upsilon,\Theta}\left(\mathsf{p}_n, \frac{1}{n+1}\right) \not\subseteq O_\lambda$$

for every $\lambda \in \Lambda$.

By Theorem 4.1, the sequence (p_n) has a neutrosophic intuitionistic fuzzy Cauchy subsequence (p_{n_j}). Completeness provides $\mathsf{p} \in \mathsf{U}$ such that

$$\mathsf{p}_{n_j} \longrightarrow \mathsf{p}.$$

Choose $\lambda_0 \in \Lambda$ with $\mathsf{p} \in O_{\lambda_0}$. Since O_{λ_0} is open, there exists $r \in (0,1)$ such that

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r) \subseteq O_{\lambda_0}.$$

Choose $\delta \in (0,1)$ satisfying

$$(1-\delta) * (1-\delta) > 1 - r$$

and

$$\delta \diamond \delta < r.$$

For all sufficiently large j,

$$\Omega_{\mathsf{U}}(\mathsf{p}_{n_j} - \mathsf{p}) > 1 - \delta,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{p}_{n_j} - \mathsf{p}) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{p}_{n_j} - \mathsf{p}) < \delta,$$

and

$$\frac{1}{n_j+1} < \delta.$$

Fix such a j, and let

$$\mathsf{L} \in B_{\Omega,\Upsilon,\Theta}\left(\mathsf{p}_{n_j}, \frac{1}{n_j+1}\right).$$

Then,

$$\Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_{n_j}) > 1 - \frac{1}{n_j+1} > 1 - \delta,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_{n_j}) < \frac{1}{n_j+1} < \delta,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{p}_{n_j}) < \frac{1}{n_j+1} < \delta.$$

Using

$$\mathsf{L} - \mathsf{p} = (\mathsf{L} - \mathsf{p}_{n_j}) + (\mathsf{p}_{n_j} - \mathsf{p}),$$

we obtain

$$\Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{p}) > (1-\delta) * (1-\delta) > 1 - r,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{p}) < \delta \diamond \delta < r,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{p}) < \delta \diamond \delta < r.$$

Therefore,

$$\mathsf{L} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r) \subseteq O_{\lambda_0}.$$

It follows that

$$B_{\Omega,\Upsilon,\Theta}\left(\mathsf{p}_{n_j}, \frac{1}{n_j+1}\right) \subseteq O_{\lambda_0},$$

contradicting the choice of this ball. Hence, $\mathcal{O}$ has a finite subcover, and U is NIAF-compact. □

Corollary 4.1. *A neutrosophic intuitionistic algebra fuzzy normed space is NIAF-compact if and only if it is NIAF-totally bounded and complete with respect to its neutrosophic intuitionistic algebra fuzzy norm.*

Proof. If U is NIAF-compact, then it is NIAF-totally bounded by Proposition 4.2 and complete by Proposition 4.3. The converse is Theorem 4.3. □

5. Neutrosophic Intuitionistic Fuzzy Limit Points and Compactness in Neutrosophic Intuitionistic Algebra Fuzzy Normed Spaces

In this section, we investigate further compactness properties in neutrosophic intuitionistic algebra fuzzy normed spaces. In particular, we introduce neutrosophic intuitionistic fuzzy limit points and use them to study the relationships among compactness, sequential compactness, closedness, and convergent subsequences. The results show that the classical limit-point and sequential characterizations of compactness remain valid when closeness is described simultaneously by the truth-membership, indeterminacy-membership, and falsity-membership functions.

Definition 5.1. *Let $(\mathsf{U}, \Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}}, *, \diamond)$ be a neutrosophic intuitionistic algebra fuzzy normed space, and let $S \subseteq \mathsf{U}$. A point $\mathsf{L} \in \mathsf{U}$ is called a* neutrosophic intuitionistic fuzzy limit point *of S, or briefly an* NIAF-limit point *of S, if, for every $r \in (0,1)$,*

$$(B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) \setminus \{\mathsf{L}\}) \cap S \neq \varnothing.$$

The set of all NIAF-limit points of S is called the NIAF-derived set *of S and is denoted by S'_{NIAF}.*

Definition 5.2. *Let $S \subseteq \mathsf{U}$ and let $\mathsf{L} \in S$. The point L is called an* NIAF-isolated point *of S if there exists $r \in (0,1)$ such that*

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) \cap S = \{\mathsf{L}\}.$$

Example 5.1. *Let $\lambda \in (0,1)$ and consider the neutrosophic intuitionistic algebra fuzzy absolute value on $\mathbb{R}$ defined by*

$$\Omega_{\mathbb{R}}(x) = \frac{1}{1+|x|}, \quad \Upsilon_{\mathbb{R}}(x) = \lambda\frac{|x|}{1+|x|}, \qquad \Theta_{\mathbb{R}}(x) = (1-\lambda)\frac{|x|}{1+|x|},$$

together with the product t-norm and the algebraic-sum t-conorm. Consider

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Now, we show that 0 is an NIAF-limit point of S.

Let $r \in (0,1)$. Choose $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \frac{r}{1-r}.$$

Then,

$$\frac{1/N}{1+1/N} < r.$$

Consequently,

$$\Omega_{\mathbb{R}}\left(\frac{1}{N}\right) > 1-r,$$

$$\Upsilon_{\mathbb{R}}\left(\frac{1}{N}\right) < r, \qquad \Theta_{\mathbb{R}}\left(\frac{1}{N}\right) < r.$$

Hence,

$$\frac{1}{N} \in B_{\Omega,\Upsilon,\Theta}(0, r) \cap S, \qquad \frac{1}{N} \neq 0.$$

Since r was arbitrary, 0 is an NIAF-limit point of S.

On the other hand, the NIAF topology induced by these three functions coincides with the ordinary topology on $\mathbb{R}$, as shown in Example 4.1. Every point $1/m$ of S is isolated in the ordinary topology and therefore also NIAF-isolated. Thus,

$$S'_{\mathrm{NIAF}} = \{0\}.$$

Lemma 5.1. *Let $S \subseteq \mathsf{U}$ and let L be an NIAF-limit point of S. Then every NIAF-open ball centred at L contains infinitely many distinct points of $S \setminus \{\mathsf{L}\}$.*

Proof. Suppose, to the contrary, that there exists $r \in (0,1)$ such that

$$(B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) \setminus \{\mathsf{L}\}) \cap S = \{s_1, s_2, \ldots, s_m\}$$

for some finite collection of distinct points. For every $i \in \{1, \ldots, m\}$, we have $s_i \neq \mathsf{L}$. By the definiteness axioms,

$$\Omega_{\mathsf{U}}(\mathsf{L} - s_i) < 1, \qquad \Upsilon_{\mathsf{U}}(\mathsf{L} - s_i) > 0, \qquad \Theta_{\mathsf{U}}(\mathsf{L} - s_i) > 0.$$

Choose $r_i \in (0,1)$ such that

$$r_i < \min\left\{1 - \Omega_{\mathsf{U}}(\mathsf{L} - s_i), \Upsilon_{\mathsf{U}}(\mathsf{L} - s_i), \Theta_{\mathsf{U}}(\mathsf{L} - s_i)\right\}.$$

Then $s_i \notin B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r_i)$. Put

$$\rho = \min\{r, r_1, r_2, \ldots, r_m\}.$$

Since $\rho \leq r$, we have

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, \rho) \subseteq B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r).$$

Moreover, none of $s_1, \ldots, s_m$ belongs to the smaller ball. Therefore,

$$(B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, \rho) \setminus \{\mathsf{L}\}) \cap S = \varnothing,$$

contradicting the assumption that L is an NIAF-limit point of S. □

Lemma 5.2. *The limit of a convergent sequence in a neutrosophic intuitionistic algebra fuzzy normed space is unique.*

Proof. Suppose that $\mathsf{L}_n \to \mathsf{L}$ and $\mathsf{L}_n \to v$. Let $\varepsilon \in (0,1)$. Choose $\delta \in (0,1)$ such that

$$(1-\delta) * (1-\delta) > 1 - \varepsilon$$

and

$$\delta \diamond \delta < \varepsilon.$$

For all sufficiently large n,

$$\Omega_{\mathsf{U}}(\mathsf{L} - \mathsf{L}_n) > 1 - \delta, \qquad \Omega_{\mathsf{U}}(\mathsf{L}_n - v) > 1 - \delta,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{L} - \mathsf{L}_n) < \delta, \qquad \Upsilon_{\mathsf{U}}(\mathsf{L}_n - v) < \delta,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - \mathsf{L}_n) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_n - v) < \delta.$$

Since

$$\mathsf{L} - v = (\mathsf{L} - \mathsf{L}_n) + (\mathsf{L}_n - v),$$

we obtain

$$\Omega_{\mathsf{U}}(\mathsf{L} - v) > (1-\delta) * (1-\delta) > 1 - \varepsilon,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{L} - v) < \delta \diamond \delta < \varepsilon,$$

and

$$\Theta_{\mathsf{U}}(\mathsf{L} - v) < \delta \diamond \delta < \varepsilon.$$

Since ε is arbitrary,

$$\Omega_{\mathsf{U}}(\mathsf{L} - v) = 1, \qquad \Upsilon_{\mathsf{U}}(\mathsf{L} - v) = 0, \qquad \Theta_{\mathsf{U}}(\mathsf{L} - v) = 0.$$

By definiteness, $\mathsf{L} = v$. □

5.1. **Limit-Point Compactness and Sequential Compactness.** Limit points and sequential compactness are closely related in fuzzy and intuitionistic fuzzy topological settings [15]. Compactness properties in fuzzy normed spaces were also studied by Deschrijver, O'Regan, Saadati, and Vaezpour [4]. Motivated by these works, we establish the corresponding results in the present three-component setting.

Definition 5.3. *A subset $P \subseteq \mathsf{U}$ is called* NIAF-sequentially compact *if every sequence in P has a subsequence that converges, in the neutrosophic intuitionistic algebra fuzzy sense, to a point of P.*

Proposition 5.1. *The following statements are equivalent:*

(1) *Every infinite subset of U has at least one NIAF-limit point in U.*
(2) *Every sequence in U has a neutrosophic intuitionistic fuzzy convergent subsequence.*

Proof. Assume first that every infinite subset of U has an NIAF-limit point, and let (L_k) be a sequence in U.

If its range

$$\{\mathsf{L}_k : k \in \mathbb{N}\}$$

is finite, then one value occurs infinitely many times. The corresponding constant subsequence converges to that value.

Suppose that the range

$$S = \{\mathsf{L}_k : k \in \mathbb{N}\}$$

is infinite. By assumption, S has an NIAF-limit point $\mathsf{L} \in \mathsf{U}$. Put

$$r_j = \frac{1}{j+1}, \qquad j \in \mathbb{N}.$$

We construct increasing indices $k_1 < k_2 < \cdots$ recursively. By Lemma 5.1, the ball $B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r_1)$ contains infinitely many distinct points of $S \setminus \{\mathsf{L}\}$. Choose one of these values and an index k_1 at which it occurs.

Suppose that $k_1 < \cdots < k_{j-1}$ have been chosen. The ball $B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r_j)$ contains infinitely many distinct points of $S \setminus \{\mathsf{L}\}$, whereas only finitely many values occur among $\mathsf{L}_1, \ldots, \mathsf{L}_{k_{j-1}}$. Hence, one can choose $k_j > k_{j-1}$ such that

$$\mathsf{L}_{k_j} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r_j).$$

It follows that

$$\Omega_{\mathsf{U}}(\mathsf{L}_{k_j} - \mathsf{L}) > 1 - \frac{1}{j+1},$$

$$\Upsilon_{\mathsf{U}}(\mathsf{L}_{k_j} - \mathsf{L}) < \frac{1}{j+1}, \qquad \Theta_{\mathsf{U}}(\mathsf{L}_{k_j} - \mathsf{L}) < \frac{1}{j+1}.$$

Therefore, $\mathsf{L}_{k_j} \to \mathsf{L}$.

Conversely, assume that every sequence in U has a convergent subsequence. Let S be an infinite subset of U. Choose a sequence (p_k) of distinct points of S. By assumption, there exist a subsequence (p_{k_j}) and a point $\mathsf{p} \in \mathsf{U}$ such that

$$\mathsf{p}_{k_j} \longrightarrow \mathsf{p}.$$

Let $r \in (0,1)$. For all sufficiently large j,

$$\mathsf{p}_{k_j} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r).$$

Since the points p_{k_j} are distinct, at most one of them can equal p. Thus, for some sufficiently large j,

$$\mathsf{p}_{k_j} \in (B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r) \setminus \{\mathsf{p}\}) \cap S.$$

Therefore, p is an NIAF-limit point of S. □

Corollary 5.1. *Every infinite subset of an NIAF-compact subset P has an NIAF-limit point belonging to P.*

Proof. Let $S \subseteq P$ be infinite and choose a sequence (p_k) of distinct points of S. By Theorem 4.2, there exist a subsequence (p_{k_j}) and a point $\mathsf{p} \in P$ such that

$$\mathsf{p}_{k_j} \longrightarrow \mathsf{p}.$$

Let $r \in (0,1)$. For all sufficiently large j, $\mathsf{p}_{k_j} \in B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r)$. Since the terms are distinct, one may choose such a term different from p. Thus,

$$(B_{\Omega,\Upsilon,\Theta}(\mathsf{p}, r) \setminus \{\mathsf{p}\}) \cap S \neq \varnothing.$$

Hence, p is an NIAF-limit point of S belonging to P. □

Theorem 5.1. *For a neutrosophic intuitionistic algebra fuzzy normed space U, the following statements are equivalent:*

(1) U *is NIAF-compact.*
(2) *Every sequence in* U *has a neutrosophic intuitionistic fuzzy convergent subsequence.*

Proof. (1) ⇒ (2). This is Theorem 4.2.

(2) ⇒ (1). Assume that every sequence in U has a neutrosophic intuitionistic fuzzy convergent subsequence.

We first prove that U is NIAF-totally bounded. Suppose, to the contrary, that it is not NIAF-totally bounded. Then there exists $r \in (0,1)$ such that U has no finite NIAF r-net. Choose $\mathsf{L}_1 \in \mathsf{U}$. After choosing $\mathsf{L}_1, \ldots, \mathsf{L}_n$, select

$$\mathsf{L}_{n+1} \in \mathsf{U} \setminus \bigcup_{j=1}^{n} B_{\Omega,\Upsilon,\Theta}(\mathsf{L}_j, r).$$

Then, whenever $m > j$,

$$\mathsf{L}_m \notin B_{\Omega,\Upsilon,\Theta}(\mathsf{L}_j, r).$$

Thus, at least one of the following holds:

$$\Omega_{\mathsf{U}}(\mathsf{L}_m - \mathsf{L}_j) \leq 1 - r, \ \Upsilon_{\mathsf{U}}(\mathsf{L}_m - \mathsf{L}_j) \geq r, \text{ or } \Theta_{\mathsf{U}}(\mathsf{L}_m - \mathsf{L}_j) \geq r.$$

Therefore, no subsequence of (L_n) is a neutrosophic intuitionistic fuzzy Cauchy sequence. Since every convergent sequence is Cauchy by Theorem 3.1, the sequence has no convergent subsequence, contradicting the assumption. Hence, U is NIAF-totally bounded.

We next prove that U is complete. Let (p_k) be a neutrosophic intuitionistic fuzzy Cauchy sequence in U. By the assumption, there exist a subsequence (p_{k_i}) and a point $\mathsf{p} \in \mathsf{U}$ such that

$$\mathsf{p}_{k_i} \longrightarrow \mathsf{p}.$$

Let $\varepsilon \in (0,1)$. Choose $\delta \in (0,1)$ satisfying

$$(1-\delta) * (1-\delta) > 1 - \varepsilon \text{ and } \delta \diamond \delta < \varepsilon.$$

Since (p_k) is Cauchy, there exists $N \in \mathbb{N}$ such that, for all $k, m \geq N$,

$$\Omega_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_m) > 1 - \delta, \ \Upsilon_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_m) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}_m) < \delta.$$

Choose i sufficiently large that $k_i \geq N$ and

$$\Omega_{\mathsf{U}}(\mathsf{p}_{k_i} - \mathsf{p}) > 1 - \delta, \ \Upsilon_{\mathsf{U}}(\mathsf{p}_{k_i} - \mathsf{p}) < \delta, \qquad \Theta_{\mathsf{U}}(\mathsf{p}_{k_i} - \mathsf{p}) < \delta.$$

For every $k \geq N$,

$$\mathsf{p}_k - \mathsf{p} = (\mathsf{p}_k - \mathsf{p}_{k_i}) + (\mathsf{p}_{k_i} - \mathsf{p}).$$

Consequently,

$$\Omega_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}) > (1-\delta) * (1-\delta) > 1 - \varepsilon,$$
$$\Upsilon_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}) < \delta \diamond \delta < \varepsilon, \text{ and } \Theta_{\mathsf{U}}(\mathsf{p}_k - \mathsf{p}) < \delta \diamond \delta < \varepsilon.$$

Thus, $\mathsf{p}_k \to \mathsf{p}$, and U is complete. By Theorem 4.3, U is NIAF-compact. □

Corollary 5.2. *Let* U *be NIAF-compact and let* P *be closed in the neutrosophic intuitionistic fuzzy topology of* U*. Then* P *is NIAF-compact.*

Proof. Let $\mathcal{O} = \{O_\lambda : \lambda \in \Lambda\}$ be a neutrosophic intuitionistic fuzzy open cover of P. Since P is closed, $\mathsf{U} \setminus P$ is open. Therefore,

$$\mathcal{O} \cup \{\mathsf{U} \setminus P\}$$

is an open cover of U. By the NIAF-compactness of U, this cover has a finite subcover. After removing $\mathsf{U} \setminus P$, if it occurs, the remaining finitely many members of $\mathcal{O}$ still cover P. Hence, P is NIAF-compact. □

The relationship between compactness and closedness is fundamental in fuzzy and intuitionistic fuzzy topological structures [15, 16]. Compactness in fuzzy normed spaces was also studied by Deschrijver, O'Regan, Saadati, and Vaezpour [4]. The next theorem extends the classical fact that compact subsets of Hausdorff spaces are closed.

Theorem 5.2. *Every NIAF-compact subset* P *of a neutrosophic intuitionistic algebra fuzzy normed space* U *is closed in the neutrosophic intuitionistic fuzzy topology of* U*.*

Proof. Let $\mathsf{p} \in \overline{P}$. By Theorem 3.4, there exists a sequence (p_k) in P such that

$$\mathsf{p}_k \longrightarrow \mathsf{p}.$$

Since P is NIAF-compact, Theorem 4.2 yields a subsequence (p_{k_j}) and a point $q \in P$ such that

$$\mathsf{p}_{k_j} \longrightarrow q.$$

Every subsequence of a convergent sequence converges to the same limit, so $\mathsf{p}_{k_j} \to \mathsf{p}$. By Lemma 5.2, $\mathsf{p} = q$. Hence, $\mathsf{p} \in P$. Therefore, $\overline{P} \subseteq P$, and the reverse inclusion is automatic. Thus, $P = \overline{P}$, so P is closed. □

Corollary 5.3. *Let* U *be NIAF-compact. A subset* $P \subseteq \mathsf{U}$ *is NIAF-compact if and only if it is closed in the neutrosophic intuitionistic fuzzy topology of* U*.*

Proof. If P is NIAF-compact, then it is closed by Theorem 5.2. Conversely, every closed subset of the NIAF-compact space U is NIAF-compact by Corollary 5.2. □

6. Equivalence and Compactness in Finite-Dimensional Neutrosophic Intuitionistic Algebra Fuzzy Normed Spaces

In this section, we study finite-dimensional neutrosophic intuitionistic algebra fuzzy normed spaces through the equivalence of their norm structures. Equivalent neutrosophic intuitionistic algebra fuzzy norms produce the same convergent sequences and, under the regularity conditions introduced below, the same Cauchy sequences, bounded sets, complete structures, and compact subsets. Consequently, the finite-dimensional properties obtained in this section do not depend on a particular choice of an equivalent neutrosophic intuitionistic algebra fuzzy norm.

The basic axioms of a neutrosophic intuitionistic algebra fuzzy norm do not, by themselves, guarantee that every such norm on a finite-dimensional space generates the ordinary finite-dimensional vector-space topology. Therefore, to formulate a mathematically valid norm-equivalence theorem and a Heine–Borel theorem, we work with the following natural class of norm-compatible neutrosophic intuitionistic algebra fuzzy norms. This class contains the standard examples obtained from ordinary algebra norms by means of continuous truth-, indeterminacy-, and falsity-profile functions.

Definition 6.1. *Let* $\mathcal{N} = (\Omega_{\mathsf{U}}, \Upsilon_{\mathsf{U}}, \Theta_{\mathsf{U}})$ *be a neutrosophic intuitionistic algebra fuzzy norm on a real algebra* U*. The norm* $\mathcal{N}$ *is called* norm-compatible *if there exist an ordinary algebra norm* $\|\cdot\|$ *on* U *and continuous functions*

$$\varphi : [0,\infty) \longrightarrow (0,1], \qquad \psi, \chi : [0,\infty) \longrightarrow [0,1)$$

such that:

(1) φ *is strictly decreasing,* $\varphi(0) = 1$, *and* $\lim_{s\to\infty} \varphi(s) = 0$;
(2) ψ *and* χ *are strictly increasing,* $\psi(0) = \chi(0) = 0$, *and* $\lim_{s\to\infty} \psi(s) = \lim_{s\to\infty} \chi(s) = 1$;
(3) *for every* $\mathsf{L} \in \mathsf{U}$,

$$\Omega_{\mathsf{U}}(\mathsf{L}) = \varphi(\|\mathsf{L}\|), \ \Upsilon_{\mathsf{U}}(\mathsf{L}) = \psi(\|\mathsf{L}\|), \qquad \Theta_{\mathsf{U}}(\mathsf{L}) = \chi(\|\mathsf{L}\|).$$

The norm $\|\cdot\|$ *is called an* associated ordinary norm, *and* φ, ψ, *and* χ *are called the corresponding* neutrosophic profile functions.

Remark 6.1. *Definition 6.1 is an additional regularity condition; it is not a consequence of the axioms in Definition 3.8. It is precisely the condition that connects the three neutrosophic components with an ordinary norm and therefore permits the use of finite-dimensional norm-equivalence, Bolzano–Weierstrass, Heine–Borel, and Riesz-type arguments.*

For a norm-compatible NIAF norm $\mathcal{N}$, define its radial deviation function by

$$\rho_{\mathcal{N}}(s) = \max\{1 - \varphi(s), \psi(s), \chi(s)\}, \qquad s \geq 0.$$

The function $\rho_{\mathcal{N}}$ is continuous and strictly increasing, $\rho_{\mathcal{N}}(0) = 0$, and $\lim_{s\to\infty} \rho_{\mathcal{N}}(s) = 1$. Hence, it has a continuous inverse

$$R_{\mathcal{N}} : (0,1) \longrightarrow (0,\infty), \qquad R_{\mathcal{N}}(r) = \rho_{\mathcal{N}}^{-1}(r).$$

Lemma 6.1. *Let* $\mathcal{N}$ *be a norm-compatible NIAF norm with associated ordinary norm* $\|\cdot\|$. *Then, for every* $\mathsf{L} \in \mathsf{U}$ *and every* $r \in (0,1)$,

$$B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) = \{v \in \mathsf{U} : \|v - \mathsf{L}\| < R_{\mathcal{N}}(r)\}.$$

Moreover,

$$\overline{B}_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r) = \{v \in \mathsf{U} : \|v - \mathsf{L}\| \leq R_{\mathcal{N}}(r)\}.$$

Proof. For $v \in \mathsf{U}$, the three inequalities

$$\Omega_{\mathsf{U}}(v - \mathsf{L}) > 1 - r, \qquad \Upsilon_{\mathsf{U}}(v - \mathsf{L}) < r, \qquad \Theta_{\mathsf{U}}(v - \mathsf{L}) < r$$

hold if and only if

$$\max\{1 - \Omega_{\mathsf{U}}(v - \mathsf{L}), \Upsilon_{\mathsf{U}}(v - \mathsf{L}), \Theta_{\mathsf{U}}(v - \mathsf{L})\} < r.$$

By norm compatibility, this is equivalent to

$$\rho_{\mathcal{N}}(\|v - \mathsf{L}\|) < r.$$

Since $\rho_{\mathcal{N}}$ is strictly increasing, the last inequality is equivalent to

$$\|v - \mathsf{L}\| < \rho_{\mathcal{N}}^{-1}(r) = R_{\mathcal{N}}(r).$$

The proof for the closed ball is identical with the strict inequalities replaced by non-strict inequalities. □

Corollary 6.1. *Let* $\mathcal{N}$ *be a norm-compatible NIAF norm with associated ordinary norm* $\|\cdot\|$. *For a sequence* (L_n) *in* U *and* $\mathsf{L} \in \mathsf{U}$,

$$\mathsf{L}_n \longrightarrow \mathsf{L} \quad \textit{in the NIAF sense if and only if } \|\mathsf{L}_n - \mathsf{L}\| \longrightarrow 0.$$

Proof. Assume first that $\mathsf{L}_n \to \mathsf{L}$ in the NIAF sense. Let $\varepsilon > 0$. Since $\rho_{\mathcal{N}}(\varepsilon) \in (0,1)$, convergence gives an index N such that

$$\rho_{\mathcal{N}}(\|\mathsf{L}_n - \mathsf{L}\|) < \rho_{\mathcal{N}}(\varepsilon)$$

for every $n \geq N$. Strict monotonicity yields $\|\mathsf{L}_n - \mathsf{L}\| < \varepsilon$.

Conversely, assume that $\|\mathsf{L}_n - \mathsf{L}\| \to 0$. Let $r \in (0,1)$. For all sufficiently large n,

$$\|\mathsf{L}_n - \mathsf{L}\| < R_{\mathcal{N}}(r).$$

By Lemma 6.1, this means that

$$\mathsf{L}_n \in B_{\Omega,\Upsilon,\Theta}(\mathsf{L}, r).$$

Hence, $\mathsf{L}_n \to \mathsf{L}$ in the NIAF sense. □

Corollary 6.2. *A sequence (L_n) is a neutrosophic intuitionistic fuzzy Cauchy sequence if and only if it is Cauchy with respect to the associated ordinary norm.*

Proof. Apply Lemma 6.1 to the differences $\mathsf{L}_n - \mathsf{L}_m$. □

Definition 6.2. *Let $\mathcal{N}_1 = (\Omega_1, \Upsilon_1, \Theta_1)$ and $\mathcal{N}_2 = (\Omega_2, \Upsilon_2, \Theta_2)$ be two neutrosophic intuitionistic algebra fuzzy norms on the same real algebra U. The norms $\mathcal{N}_1$ and $\mathcal{N}_2$ are called* equivalent *if, for every sequence (L_k) in U and every $\mathsf{L} \in \mathsf{U}$,*

$$\mathsf{L}_k \longrightarrow \mathsf{L} \quad \text{with respect to } \mathcal{N}_1 \text{ if and only if } \mathsf{L}_k \longrightarrow \mathsf{L} \quad \text{with respect to } \mathcal{N}_2.$$

Example 6.1. *Let $\lambda \in (0,1)$ and let $\mathsf{U} = \mathbb{R}$. For $c \geq 1$, define*

$$\Omega_c(x) = \frac{1}{1+c|x|}, \ \Upsilon_c(x) = \lambda \frac{c|x|}{1+c|x|}, \qquad \Theta_c(x) = (1-\lambda)\frac{c|x|}{1+c|x|}.$$

With the product t-norm and the algebraic-sum t-conorm, these functions give the standard norm-compatible NIAF structures associated with the usual absolute value. Consider the cases $c = 1$ and $c = 2$. For a sequence (x_n) in $\mathbb{R}$ and $x \in \mathbb{R}$,

$$\Omega_c(x_n - x) \longrightarrow 1, \quad \Upsilon_c(x_n - x) \longrightarrow 0, \quad \Theta_c(x_n - x) \longrightarrow 0$$

if and only if

$$|x_n - x| \longrightarrow 0.$$

Therefore,

$$(\Omega_1, \Upsilon_1, \Theta_1) \quad \text{and} \quad (\Omega_2, \Upsilon_2, \Theta_2)$$

are equivalent NIAF norms on $\mathbb{R}$.

The comparison of two fuzzy norms on the same vector space is useful for identifying when both structures generate the same convergence behaviour. Related finite-dimensional and topological ideas in fuzzy normed spaces were developed in [5, 16, 18]. The following theorem gives a valid comparison criterion for norm-compatible NIAF norms.

Theorem 6.1. *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be norm-compatible NIAF norms on U, with associated ordinary norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Suppose that there exist constants $a, b > 0$ such that*

$$a\|\mathsf{L}\|_2 \leq \|\mathsf{L}\|_1 \leq b\|\mathsf{L}\|_2$$

for every $\mathsf{L} \in \mathsf{U}$. Then $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent.

Proof. Let (L_k) be a sequence in U and let $\mathsf{L} \in \mathsf{U}$. Suppose that $\mathsf{L}_k \to \mathsf{L}$ with respect to $\mathcal{N}_1$. By Corollary 6.1,

$$\|\mathsf{L}_k - \mathsf{L}\|_1 \longrightarrow 0.$$

Since

$$\|\mathsf{L}_k - \mathsf{L}\|_2 \leq \frac{1}{a}\|\mathsf{L}_k - \mathsf{L}\|_1,$$

we obtain $\|\mathsf{L}_k - \mathsf{L}\|_2 \to 0$. Another application of Corollary 6.1 yields convergence with respect to $\mathcal{N}_2$.

Conversely, if $\mathsf{L}_k \to \mathsf{L}$ with respect to $\mathcal{N}_2$, then $\|\mathsf{L}_k - \mathsf{L}\|_2 \to 0$. The inequality

$$\|\mathsf{L}_k - \mathsf{L}\|_1 \leq b\|\mathsf{L}_k - \mathsf{L}\|_2$$

implies convergence in $\|\cdot\|_1$, and hence convergence with respect to $\mathcal{N}_1$. Thus, the two NIAF norms are equivalent. □

Proposition 6.1. *Under the assumptions of Theorem 6.1, the two NIAF norms have the same convergent sequences, Cauchy sequences, bounded subsets, totally bounded subsets, complete subspaces, and compact subsets.*

Proof. Convergent and Cauchy sequences correspond to convergent and Cauchy sequences in the associated ordinary norms by Corollaries 6.1 and 6.2. Equivalent ordinary norms have the same convergent and Cauchy sequences, the same bounded and totally bounded sets, and the same complete subspaces. By Lemma 6.1, the NIAF topologies are the same as the corresponding ordinary norm topologies. Therefore, the compact subsets also coincide. □

6.1. **Finite-Dimensional Equivalence.** In finite-dimensional spaces, equivalence of norm structures ensures that convergence is independent of the chosen norm. Finite-dimensional and topological properties of fuzzy normed spaces have been studied in [5, 16, 18]. We now obtain the corresponding result for norm-compatible NIAF norms.

Theorem 6.2. *Let* U *be a finite-dimensional real algebra. Any two norm-compatible neutrosophic intuitionistic algebra fuzzy norms on* U *are equivalent.*

Proof. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be norm-compatible NIAF norms, with associated ordinary norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Since U is finite dimensional, the classical finite-dimensional norm-equivalence theorem gives constants $a, b > 0$ such that

$$a\|\mathsf{L}\|_2 \leq \|\mathsf{L}\|_1 \leq b\|\mathsf{L}\|_2$$

for every $\mathsf{L} \in \mathsf{U}$. The conclusion follows from Theorem 6.1. □

Theorem 6.3. *Let* U *be finite dimensional, let* $\mathcal{B} = \{\mathsf{e}_1, \mathsf{e}_2, \ldots, \mathsf{e}_k\}$ *be a basis, and let* $\mathcal{N}$ *be a norm-compatible NIAF norm on* U*. Suppose that*

$$\mathsf{L}_m = \sum_{j=1}^{k} \alpha_j^{(m)} \mathsf{e}_j \quad \textit{and} \quad \mathsf{L} = \sum_{j=1}^{k} \alpha_j \mathsf{e}_j.$$

Then,

$$\mathsf{L}_m \longrightarrow \mathsf{L} \quad \textit{in the NIAF sense if and only if } \alpha_j^{(m)} \longrightarrow \alpha_j$$

for every $j \in \{1, 2, \ldots, k\}$.

Proof. Let $\|\cdot\|$ be an associated ordinary norm. In a finite-dimensional vector space, the coordinate map

$$T : \mathbb{R}^k \longrightarrow \mathsf{U}, \qquad T(\alpha_1, \ldots, \alpha_k) = \sum_{j=1}^{k} \alpha_j \mathsf{e}_j,$$

is a linear homeomorphism. Hence,

$$\|\mathsf{L}_m - \mathsf{L}\| \longrightarrow 0$$

if and only if $\alpha_j^{(m)} \to \alpha_j$ for every j. The result follows from Corollary 6.1. □

Lemma 6.2. *Let* $\mathcal{N}$ *be a norm-compatible NIAF norm on* U*, with associated ordinary norm* $\|\cdot\|$*. A nonempty subset* $D \subseteq \mathsf{U}$ *is NIAF-bounded in the sense of Definition 4.4 if and only if it is bounded with respect to* $\|\cdot\|$*.*

Proof. Assume first that D is NIAF-bounded. Then there exists $r \in (0, 1)$ such that, for every $x, y \in D$,

$$\rho_{\mathcal{N}}(\|x - y\|) < r.$$

Therefore,

$$\|x - y\| < R_{\mathcal{N}}(r)$$

for all $x, y \in D$. Fixing $y_0 \in D$ gives

$$\|x\| \leq \|x - y_0\| + \|y_0\| < R_{\mathcal{N}}(r) + \|y_0\|$$

for every $x \in D$. Thus, D is ordinarily bounded.

Conversely, suppose that D is bounded. Then there exists $M > 0$ such that $\|x\| \leq M$ for every $x \in D$. Hence,

$$\|x - y\| \leq 2M$$

for all $x, y \in D$. Since $\rho_{\mathcal{N}}(2M) < 1$, choose $r \in (0, 1)$ such that

$$\rho_{\mathcal{N}}(2M) < r < 1.$$

Then

$$\rho_{\mathcal{N}}(\|x - y\|) < r$$

for every $x, y \in D$, which is exactly the NIAF-boundedness of D. □

6.2. **A Heine–Borel Theorem.** Inspired by the finite-dimensional fuzzy normed-space results of Felbin [5], we obtain the following Heine–Borel characterization for norm-compatible NIAF spaces.

Theorem 6.4. *Let U be a finite-dimensional norm-compatible NIAFNS and let $D \subseteq \mathsf{U}$. Then the following statements are equivalent:*

(1) *D is NIAF-compact;*
(2) *D is closed in the NIAF topology and NIAF-bounded.*

Proof. (1) ⇒ (2). If D is NIAF-compact, then it is NIAF-closed by Theorem 5.2. Let $\|\cdot\|$ be an associated ordinary norm. By Lemma 6.1, the NIAF topology and the ordinary norm topology coincide. Hence, D is compact in the ordinary norm topology and is therefore ordinarily bounded. By Lemma 6.2, D is NIAF-bounded.

(2) ⇒ (1). Let $\|\cdot\|$ be an associated ordinary norm. By Lemma 6.2, the set D is bounded in $\|\cdot\|$. By Lemma 6.1, the NIAF topology agrees with the ordinary norm topology; hence, D is also closed in the ordinary norm topology. Since U is finite dimensional, the classical Heine–Borel theorem shows that D is compact in the ordinary norm topology. The two topologies are identical, so D is NIAF-compact. □

Corollary 6.3. *Every finite-dimensional norm-compatible NIAFNS is complete with respect to its neutrosophic intuitionistic algebra fuzzy norm.*

Proof. Let (L_n) be a neutrosophic intuitionistic fuzzy Cauchy sequence. By Corollary 6.2, it is Cauchy with respect to the associated ordinary norm. Every finite-dimensional normed space is complete, so there exists $\mathsf{L} \in \mathsf{U}$ such that $\|\mathsf{L}_n - \mathsf{L}\| \to 0$. By Corollary 6.1, $\mathsf{L}_n \to \mathsf{L}$ in the NIAF sense. □

6.3. **A Riesz-Type Separation Lemma.** A Riesz-type separation lemma for fuzzy normed linear spaces was considered by Bag and Samanta [3, Lemma 5.1]. The following result is the corresponding valid formulation for norm-compatible NIAF spaces.

Lemma 6.3. *Let U be a norm-compatible NIAFNS with associated ordinary norm $\|\cdot\|$ and profile functions φ, ψ, and χ. Let D be a proper NIAF-closed linear subspace of U. For every $\eta \in (0, 1)$, there exists $z \in \mathsf{U}$ such that*

$$\|z\| = 1$$

and, for every $d \in D$,

$$\|z - d\| > \eta.$$

Consequently, for every $d \in D$,

$$\Omega_{\mathsf{U}}(z - d) < \varphi(\eta), \ \Upsilon_{\mathsf{U}}(z - d) > \psi(\eta), \qquad \Theta_{\mathsf{U}}(z - d) > \chi(\eta).$$

Proof. By Lemma 6.1, the NIAF topology agrees with the topology generated by $\|\cdot\|$. Hence, D is closed in the ordinary norm topology. Choose $x \in \mathsf{U} \setminus D$ and put

$$\delta = \inf\{\|x - d\| : d \in D\}.$$

Since D is closed and $x \notin D$, we have $\delta > 0$. Because $\eta \in (0,1)$, there exists $d_0 \in D$ such that

$$\|x - d_0\| < \frac{\delta}{\eta}.$$

Define by

$$z = \frac{x - d_0}{\|x - d_0\|}.$$

Then, $\|z\| = 1$. Let $d \in D$. Since D is a linear subspace,

$$d_0 + \|x - d_0\| d \in D.$$

Therefore,

$$\begin{aligned} \|z - d\| &= \frac{\|x - d_0 - \|x - d_0\| d\|}{\|x - d_0\|} \\ &= \frac{\|x - (d_0 + \|x - d_0\| d)\|}{\|x - d_0\|} \\ &\geq \frac{\delta}{\|x - d_0\|} \\ &> \eta. \end{aligned}$$

Since φ is strictly decreasing and ψ and χ are strictly increasing, the three stated component inequalities follow immediately. □

Remark 6.2. *The parameter η in Lemma 6.3 is arbitrary because the vector is normalized after choosing a point outside the closed subspace. Merely choosing an arbitrary point outside D would only produce one neighbourhood disjoint from D and would not prove separation for every prescribed radius.*

6.4. **A Compactness Criterion for Finite Dimensionality.** The classicalA compactness criterion for finite dimensionality in fuzzy normed linear spaces was established by Bag and Samanta [3, Theorem 5.2]. Motivated by this result, we establish the corresponding compactness criterion for norm-compatible NIAF spaces. For $r \in (0,1)$, denote the closed NIAF ball centered at zero by

$$\overline{B}_{\mathcal{N}}(0,r) = \overline{B}_{\Omega,\Upsilon,\Theta}(0,r).$$

Theorem 6.5. *Let U be a norm-compatible NIAFNS. The following statements are equivalent:*

(1) *U is finite dimensional;*
(2) *every closed NIAF ball in U is NIAF-compact;*
(3) *there exists $r_0 \in (0,1)$ such that $\overline{B}_{\mathcal{N}}(0,r_0)$ is NIAF-compact.*

Proof. (1) ⇒ (2). Let $\overline{B}_{\mathcal{N}}(\mathsf{L},r)$ be a closed NIAF ball. By Lemma 6.1,

$$\overline{B}_{\mathcal{N}}(\mathsf{L},r) = \{v \in \mathsf{U} : \|v - \mathsf{L}\| \leq R_{\mathcal{N}}(r)\}.$$

It is therefore closed and bounded. By Theorem 6.4, it is NIAF-compact.

(2) ⇒ (3) is immediate.

(3) ⇒ (1). Suppose, to the contrary, that U is infinite dimensional. Put

$$R = R_{\mathcal{N}}(r_0) > 0.$$

Choose $u_1 \in \mathsf{U}$ with $\|u_1\| = 1$ and set $D_1 = \operatorname{span}\{u_1\}$. Every finite-dimensional subspace is closed in the ordinary norm topology and hence NIAF-closed. Apply Lemma 6.3 with $\eta = 1/2$ to obtain $u_2 \in \mathsf{U}$ such that

$$\|u_2\| = 1 \quad \text{and} \quad \|u_2 - d\| > \frac{1}{2}$$

for every $d \in D_1$.

Proceeding inductively, after choosing $u_1, \ldots, u_n$, let

$$D_n = \operatorname{span}\{u_1, \ldots, u_n\}.$$

Since U is infinite dimensional, D_n is a proper closed subspace. The Riesz-type lemma gives u_{n+1} such that

$$\|u_{n+1}\| = 1 \quad \text{and} \quad \|u_{n+1} - d\| > \frac{1}{2}$$

for every $d \in D_n$. In particular, whenever $m < n$,

$$\|u_n - u_m\| > \frac{1}{2}.$$

Set

$$x_n = Ru_n.$$

Then, $\|x_n\| = R$, so

$$x_n \in \overline{B}_{\mathcal{N}}(0, r_0)$$

for every n. Moreover, for $m \neq n$,

$$\|x_n - x_m\| > \frac{R}{2}.$$

Hence,

$$\rho_{\mathcal{N}}(\|x_n - x_m\|) > \rho_{\mathcal{N}}\left(\frac{R}{2}\right) > 0.$$

Choose $\varepsilon \in (0, 1)$ satisfying

$$0 < \varepsilon < \rho_{\mathcal{N}}\left(\frac{R}{2}\right).$$

No two distinct terms of (x_n) are within the NIAF ε-ball of one another. Consequently, no subsequence of (x_n) is a neutrosophic intuitionistic fuzzy Cauchy sequence and no subsequence can converge. This contradicts the NIAF-compactness of $\overline{B}_{\mathcal{N}}(0, r_0)$ and Theorem 4.2. Therefore, U is finite dimensional. □

Corollary 6.4. *A norm-compatible NIAFNS U is finite dimensional if and only if one, equivalently every, nontrivial closed NIAF ball centered at zero is NIAF-compact.*

Proof. This is an immediate consequence of Theorem 6.5. □

Remark 6.3. *A set defined only by $0 < \Omega_{\mathsf{U}}(\mathsf{L}) \leq 1$, $0 \leq \Upsilon_{\mathsf{U}}(\mathsf{L}) < 1$, $0 \leq \Theta_{\mathsf{U}}(\mathsf{L}) < 1$ need not be a bounded ball and, in a nondegenerate NIAFNS, may coincide with the entire space. It therefore cannot serve as a general compactness criterion for finite dimensionality. The correct object is a nontrivial closed NIAF ball, as used in Theorem 6.5.*

7. Application to Best Approximation

Best approximation theory is concerned with finding an element of a prescribed set that minimizes the deviation from a given point. In fuzzy analysis, Vaezpour and Karimi [62] introduced the notion of (t)-best approximation in fuzzy normed spaces and investigated (t)-proximinal and (t)-Chebyshev sets. Goudarzi and Vaezpour [63] subsequently developed simultaneous best approximation in fuzzy normed spaces, including its relationship with quotient spaces. In the intuitionistic fuzzy setting, Kavikumar, Khamis, and Manian [64] formulated (t)-best approximation using membership and non-membership degrees. Related notions of (t)-proximinality, approximative compactness, and best proximity points in fuzzy metric spaces were further examined by Abbasi and Mottaghi Golshan [65].

The preceding approaches are based either on a single fuzzy membership function or on a membership–non-membership pair. In a neutrosophic intuitionistic algebra fuzzy normed space, however, approximation must account simultaneously for the truth-membership, indeterminacy-membership, and falsity-membership degrees. The finite-dimensional compactness results established in this paper provide a natural basis for such an approximation theory. In particular, the Heine–Borel property ensures that every nonempty closed and bounded feasible set is compact, thereby allowing the existence of a best approximant to be obtained from the continuity of an appropriate neutrosophic approximation functional. To preserve the contributions of the three independent components, we introduce the following weighted approximation error.

Let $(\mathsf{X}, \mathcal{T}, \mathcal{I}, \mathcal{F}, *, \diamond)$ be a finite-dimensional neutrosophic intuitionistic algebra fuzzy normed space. For $x, y \in \mathsf{X}$, $t > 0$, and positive weights $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$\lambda_1 + \lambda_2 + \lambda_3 = 1,$$

define the scalarized approximation error by

$$\mathcal{E}_{t,\lambda}(x, y) = \lambda_1\big(1 - \mathcal{T}(x - y, t)\big) + \lambda_2\mathcal{I}(x - y, t) + \lambda_3\mathcal{F}(x - y, t).$$

Thus, a good approximation of x by y corresponds simultaneously to a large truth-membership value together with small indeterminacy- and falsity-membership values.

Definition 7.1. *Let A be a nonempty subset of X. An element $a_0 \in A$ is called a* weighted neutrosophic best approximant *to $x \in \mathsf{X}$ at level $t > 0$ if*

$$\mathcal{E}_{t,\lambda}(x, a_0) = \inf_{a \in A} \mathcal{E}_{t,\lambda}(x, a).$$

The corresponding set of weighted neutrosophic best approximants is denoted by

$$P_A^{t,\lambda}(x) = \left\{ a_0 \in A : \mathcal{E}_{t,\lambda}(x, a_0) = \inf_{a \in A} \mathcal{E}_{t,\lambda}(x, a) \right\}.$$

Example 7.1. *Let $\mathsf{X} = \mathbb{R}$, and for $y \in \mathbb{R}$ and $t > 0$, define the truth-membership, indeterminacy-membership, and falsity-membership functions by*

$$T(y, t) = \frac{t}{t + |y|}, \qquad I(y, t) = F(y, t) = \frac{|y|}{2(t + |y|)}.$$

Consider the weight vector

$$\lambda = (\lambda_T, \lambda_I, \lambda_F) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right),$$

and define

$$\mathcal{E}_{t,\lambda}(x, a) = \lambda_T\big(1 - T(x - a, t)\big) + \lambda_I I(x - a, t) + \lambda_F F(x - a, t).$$

Now, we take

$$A = [-1, 1], \qquad x = 2, \qquad t = 1.$$

For every $a \in A$, put $r = |2 - a| = 2 - a$. Then,

$$\begin{aligned}\mathcal{E}_{1,\lambda}(2, a) &= \frac{1}{2}\frac{r}{1+r} + \frac{1}{3}\frac{r}{2(1+r)} + \frac{1}{6}\frac{r}{2(1+r)} \\ &= \frac{3}{4}\frac{r}{1+r} \\ &= \frac{3(2-a)}{4(3-a)}.\end{aligned}$$

Since the function

$$r \longmapsto \frac{r}{1+r}$$

is strictly increasing on $[0, \infty)$, minimizing $\mathcal{E}_{1,\lambda}(2, a)$ is equivalent to minimizing $|2 - a|$ over $a \in [-1, 1]$. The unique minimizer is therefore $a_0 = 1$. Moreover,

$$\inf_{a \in A} \mathcal{E}_{1,\lambda}(2, a) = \mathcal{E}_{1,\lambda}(2, 1) = \frac{3}{8}.$$

Hence,

$$P_A^{1,\lambda}(2) = \{1\}.$$

Notice that uniqueness may fail when the feasible set is not convex. Indeed, if $B = \{1, 3\}$, then

$$|2 - 1| = |2 - 3| = 1,$$

and consequently

$$\mathcal{E}_{1,\lambda}(2, 1) = \mathcal{E}_{1,\lambda}(2, 3) = \frac{3}{8}.$$

Thus,

$$P_B^{1,\lambda}(2) = \{1, 3\}.$$

Theorem 7.1 (Existence of Weighted Neutrosophic Best Approximants)**.** *Let A be a nonempty closed and bounded subset of a finite-dimensional neutrosophic intuitionistic algebra fuzzy normed space X. Assume that, for every $t > 0$, the mappings*

$$z \longmapsto \mathcal{T}(z, t), \qquad z \longmapsto \mathcal{I}(z, t), \qquad z \longmapsto \mathcal{F}(z, t)$$

are continuous. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ satisfy

$$\lambda_i > 0, \qquad i = 1, 2, 3, \qquad \text{and} \qquad \lambda_1 + \lambda_2 + \lambda_3 = 1.$$

Then, for every $x \in \mathsf{X}$ and $t > 0$,

$$P_A^{t,\lambda}(x) \neq \varnothing.$$

Moreover, $P_A^{t,\lambda}(x)$ is compact.

Proof. Fix $x \in \mathsf{X}$, $t > 0$, and a positive weight vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$.

Since X is finite-dimensional and A is closed and bounded, the Heine–Borel theorem established previously implies that A is compact.

Define the function

$$g_{x,t,\lambda} \colon A \longrightarrow \mathbb{R}$$

by

$$\begin{aligned}g_{x,t,\lambda}(a) &:= \mathcal{E}_{t,\lambda}(x, a) \\ &= \lambda_1\big(1 - \mathcal{T}(x - a, t)\big) + \lambda_2\mathcal{I}(x - a, t) + \lambda_3\mathcal{F}(x - a, t).\end{aligned}$$

We first show that $g_{x,t,\lambda}$ is continuous on A. The mapping

$$a \longmapsto x - a$$

is continuous because vector subtraction is continuous in X. By assumption, for the fixed value $t > 0$, the mappings

$$z \longmapsto \mathcal{T}(z,t), \qquad z \longmapsto \mathcal{I}(z,t), \qquad z \longmapsto \mathcal{F}(z,t)$$

are continuous. Therefore, the compositions

$$a \longmapsto \mathcal{T}(x-a,t), \qquad a \longmapsto \mathcal{I}(x-a,t), \qquad a \longmapsto \mathcal{F}(x-a,t)$$

are continuous on A. Since a finite linear combination of continuous real-valued functions is continuous, it follows that $g_{x,t,\lambda}$ is continuous on A.

Because A is nonempty and compact, and $g_{x,t,\lambda}$ is continuous, the extreme value theorem guarantees the existence of an element $a_0 \in A$ such that

$$g_{x,t,\lambda}(a_0) = \min_{a\in A} g_{x,t,\lambda}(a).$$

Equivalently,

$$\mathcal{E}_{t,\lambda}(x,a_0) = \min_{a\in A} \mathcal{E}_{t,\lambda}(x,a).$$

By the definition of $P_A^{t,\lambda}(x)$, this means that

$$a_0 \in P_A^{t,\lambda}(x).$$

Hence,

$$P_A^{t,\lambda}(x) \neq \varnothing.$$

It remains to prove that $P_A^{t,\lambda}(x)$ is compact. Set

$$m_{x,t,\lambda} := \min_{a\in A} g_{x,t,\lambda}(a) = \min_{a\in A} \mathcal{E}_{t,\lambda}(x,a).$$

Then the set of all weighted neutrosophic best approximants can be written as

$$\begin{aligned} P_A^{t,\lambda}(x) &= \{a \in A : g_{x,t,\lambda}(a) = m_{x,t,\lambda}\} \\ &= g_{x,t,\lambda}^{-1}\big(\{m_{x,t,\lambda}\}\big). \end{aligned}$$

More explicitly,

$$\begin{aligned} P_A^{t,\lambda}(x) = \Bigg\{ & y \in A : \lambda_1\big(1-\mathcal{T}(x-y,t)\big) + \lambda_2\mathcal{I}(x-y,t) + \lambda_3\mathcal{F}(x-y,t) \\ & = \min_{a\in A}\Big[\lambda_1\big(1-\mathcal{T}(x-a,t)\big) + \lambda_2\mathcal{I}(x-a,t) + \lambda_3\mathcal{F}(x-a,t)\Big]\Bigg\}. \end{aligned}$$

The singleton

$$\{m_{x,t,\lambda}\}$$

is a closed subset of $\mathbb{R}$. Since $g_{x,t,\lambda}$ is continuous, its inverse image

$$g_{x,t,\lambda}^{-1}\big(\{m_{x,t,\lambda}\}\big)$$

is closed in A. Thus, $P_A^{t,\lambda}(x)$ is a closed subset of the compact set A. Therefore,

$$P_A^{t,\lambda}(x)$$

is compact. □

Example 7.2. *Let $\mathsf{X} = \mathbb{R}^2$ be equipped with the Euclidean norm $\|\cdot\|_2$. For $z \in \mathbb{R}^2$ and $t > 0$, define*

$$\mathcal{T}(z,t) = \frac{t}{t+\|z\|_2}, \qquad \mathcal{I}(z,t) = \mathcal{F}(z,t) = \frac{\|z\|_2}{2(t+\|z\|_2)}.$$

These mappings are continuous with respect to z for every fixed $t > 0$.

Consider the unit circle

$$A = \{(\cos\theta, \sin\theta) : 0 \le \theta \le 2\pi\}.$$

The set A is nonempty, closed, and bounded in the finite-dimensional space $\mathbb{R}^2$. Therefore, A is compact.

Choose

$$\lambda = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right) \qquad \text{and} \qquad t = 1.$$

For $x \in \mathbb{R}^2$ and $a \in A$, put

$$r = \|x - a\|_2.$$

Then

$$\begin{aligned}\mathcal{E}_{1,\lambda}(x, a) &= \frac{1}{2}\left(1 - \frac{1}{1+r}\right) + \frac{1}{3}\frac{r}{2(1+r)} + \frac{1}{6}\frac{r}{2(1+r)} \\ &= \frac{3}{4}\frac{r}{1+r}.\end{aligned}$$

Since the function

$$r \longmapsto \frac{r}{1+r}$$

is strictly increasing on $[0, \infty)$, minimizing $\mathcal{E}_{1,\lambda}(x, a)$ over A is equivalent to minimizing the Euclidean distance $\|x - a\|_2$.

First, take

$$x = (2, 0).$$

For

$$a_\theta = (\cos\theta, \sin\theta) \in A,$$

we obtain

$$\begin{aligned}\|x - a_\theta\|_2 &= \sqrt{(2 - \cos\theta)^2 + \sin^2\theta} \\ &= \sqrt{5 - 4\cos\theta}.\end{aligned}$$

This quantity is minimized when $\cos\theta = 1$, corresponding to

$$a_0 = (1, 0).$$

Consequently,

$$\min_{a \in A} \mathcal{E}_{1,\lambda}(x, a) = \mathcal{E}_{1,\lambda}\big((2,0),(1,0)\big) = \frac{3}{8},$$

and hence

$$P_A^{1,\lambda}\big((2,0)\big) = \{(1,0)\}.$$

Thus, the set of weighted neutrosophic best approximants is nonempty and compact.

For a second situation, take

$$x = (0, 0).$$

Every $a \in A$ satisfies

$$\|x - a\|_2 = 1.$$

Therefore,

$$\mathcal{E}_{1,\lambda}(x, a) = \frac{3}{8} \qquad \text{for every } a \in A.$$

It follows that

$$P_A^{1,\lambda}\big((0,0)\big) = A.$$

In this case, the set of weighted neutrosophic best approximants is infinite, but it remains compact, as asserted by the theorem.

The two situations in Example 7.2 are illustrated in Figure 1.

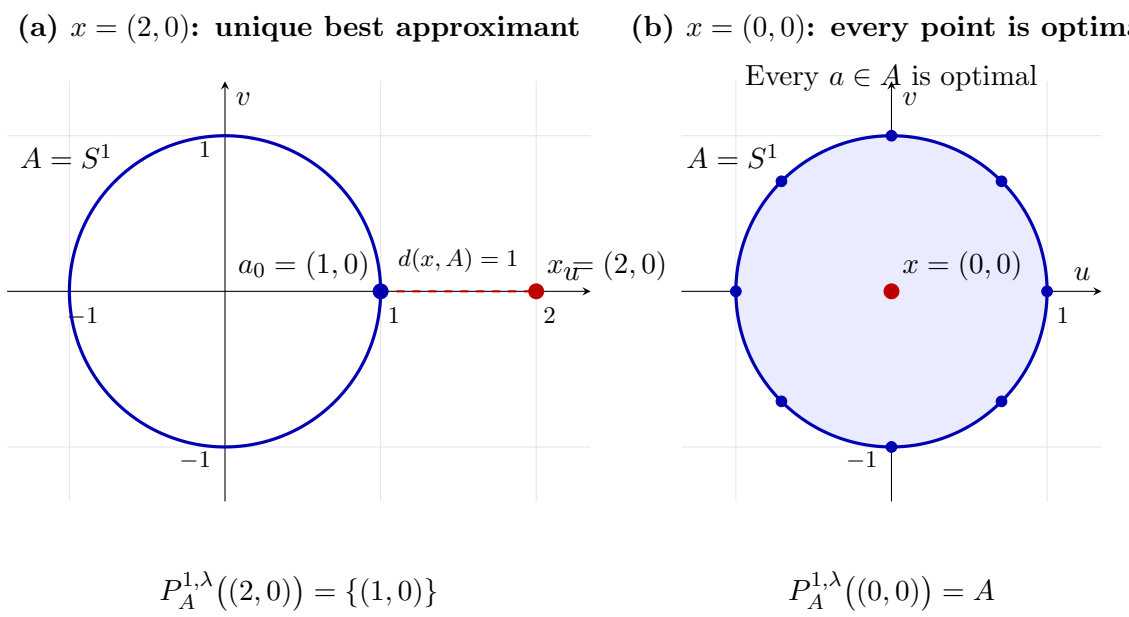

FIGURE 1. Weighted neutrosophic best approximation on the unit circle. Panel (a) shows that $a_0 = (1,0)$ is the unique best approximant to $x = (2,0)$. Panel (b) shows that every point of A is a best approximant to $x = (0,0)$; hence the approximant set is infinite but compact.

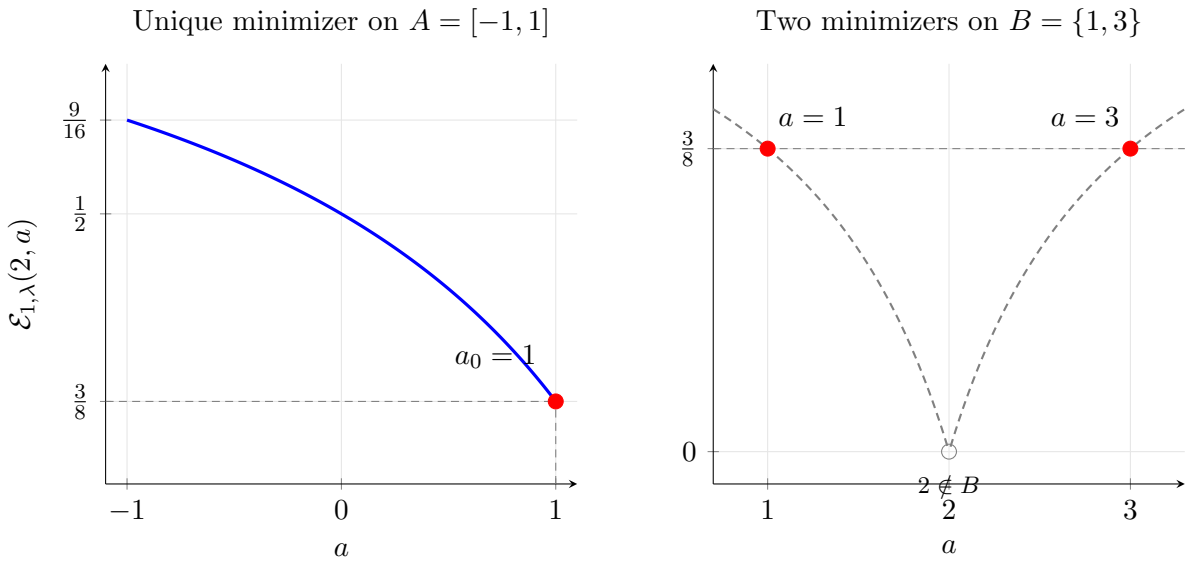

FIGURE 2. The weighted approximation error $\mathcal{E}_{1,\lambda}(2,a)$. On the convex set $A = [-1,1]$, the unique minimizer is $a_0 = 1$. On the nonconvex set $B = \{1,3\}$, both feasible points attain the minimum value $3/8$ for Example 7.1.

The weighted approximation errors over the sets A and B are illustrated in Figure 2.

The preceding theorem also admits a natural multi-objective interpretation. Associate with each $a \in A$ the error vector

$$\mathbf{e}_t(x,a) = \big(1 - \mathcal{T}(x-a,t),\, \mathcal{I}(x-a,t),\, \mathcal{F}(x-a,t)\big).$$

Corollary 7.1 (Existence of Pareto-Efficient Approximants)**.** *Under the assumptions of the preceding theorem, for every $x \in \mathsf{X}$ and every $t > 0$, the set A contains a Pareto-efficient approximant with respect to the three error components.*

Proof. Let

$$a_0 \in P_A^{t,\lambda}(x),$$

so that

$$\mathcal{E}_{t,\lambda}(x, a_0) = \min_{y \in A} \mathcal{E}_{t,\lambda}(x, y).$$

Write

$$\mathbf{e}_t(x, a) = \big(e_1(x, a), e_2(x, a), e_3(x, a)\big),$$

where

$$e_1(x, a) = 1 - \mathcal{T}(x - a, t), \qquad e_2(x, a) = \mathcal{I}(x - a, t), \qquad e_3(x, a) = \mathcal{F}(x - a, t).$$

Suppose, to the contrary, that a_0 is not Pareto-efficient. Then there exists $a \in A$ such that

$$e_i(x, a) \leq e_i(x, a_0), \qquad i = 1, 2, 3,$$

and at least one of these inequalities is strict.

Since

$$\lambda_i > 0, \qquad i = 1, 2, 3,$$

it follows that

$$\sum_{i=1}^{3} \lambda_i e_i(x, a) < \sum_{i=1}^{3} \lambda_i e_i(x, a_0).$$

Equivalently,

$$\mathcal{E}_{t,\lambda}(x, a) < \mathcal{E}_{t,\lambda}(x, a_0),$$

which contradicts the minimality of a_0. Therefore, a_0 is Pareto-efficient. □

Corollary 7.2 (Uniqueness under Strict Convexity)**.** *Suppose that A is convex and that the mapping*

$$a \longmapsto \mathcal{E}_{t,\lambda}(x, a)$$

is strictly convex on A. Then the weighted neutrosophic best approximant is unique.

Proof. Fix $x \in \mathsf{X}$, $t > 0$, and a positive weight vector λ. Let

$$m_{t,\lambda}(x) := \min_{a \in A} \mathcal{E}_{t,\lambda}(x, a).$$

Assume, to the contrary, that there exist two distinct minimizers $a_1, a_2 \in A$. Then

$$\mathcal{E}_{t,\lambda}(x, a_1) = \mathcal{E}_{t,\lambda}(x, a_2) = m_{t,\lambda}(x).$$

Since A is convex, its midpoint belongs to A; that is,

$$\frac{a_1 + a_2}{2} \in A.$$

Because the mapping

$$a \longmapsto \mathcal{E}_{t,\lambda}(x, a)$$

is strictly convex on A, we have

$$\begin{aligned} \mathcal{E}_{t,\lambda}\left(x, \frac{a_1 + a_2}{2}\right) &< \frac{1}{2}\mathcal{E}_{t,\lambda}(x, a_1) + \frac{1}{2}\mathcal{E}_{t,\lambda}(x, a_2) \\ &= \frac{1}{2}m_{t,\lambda}(x) + \frac{1}{2}m_{t,\lambda}(x) \\ &= m_{t,\lambda}(x). \end{aligned}$$

On the other hand, since $(a_1 + a_2)/2 \in A$ and $m_{t,\lambda}(x)$ is the minimum value over A, we must have

$$\mathcal{E}_{t,\lambda}\left(x, \frac{a_1 + a_2}{2}\right) \geq m_{t,\lambda}(x),$$

which is a contradiction. Therefore, $P_A^{t,\lambda}(x)$ contains exactly one element, and hence the weighted neutrosophic best approximant is unique. □

8. Concluding Remarks

In this paper, we introduced and investigated the finite-dimensional structure of neutrosophic intuitionistic algebra fuzzy normed spaces. The proposed framework is described by three functions representing the truth-membership, indeterminacy-membership, and falsity-membership degrees, together with a continuous t-norm and a continuous t-conorm. It extends ideas originating from fuzzy normed linear spaces and intuitionistic fuzzy normed spaces [5, 15, 16] by explicitly incorporating an indeterminacy component into the algebraic and topological structure.

We developed the fundamental notions of convergence, Cauchy sequences, completeness, boundedness, neighbourhoods, open and closed balls, interior points, adherent points, closure, density, limit points, total boundedness, and compactness. The resulting topology extends the topological approaches used in fuzzy and intuitionistic fuzzy normed spaces [15, 16]. In particular, we proved that every convergent sequence is Cauchy and obtained sequential characterizations of closure, density, total boundedness, and compactness.

The relationships among compactness, total boundedness, and completeness were also studied. We proved that every compact neutrosophic intuitionistic algebra fuzzy normed space is totally bounded and complete. Conversely, every complete and totally bounded space is compact. Hence,

$$\text{compactness} \iff \text{total boundedness and completeness.}$$

This result is consistent with earlier compactness principles established in intuitionistic fuzzy and L-fuzzy normed settings [15, 4]. We further showed that compact subsets are closed and that closed subsets of compact spaces are compact.

For the finite-dimensional case, we introduced an appropriate norm-compatibility condition and proved that any two compatible neutrosophic intuitionistic algebra fuzzy norms on the same finite-dimensional vector space generate the same convergence and topological structures. This extends the classical finite-dimensional fuzzy norm-equivalence theory developed by Felbin and subsequently studied in related fuzzy normed-space frameworks [5, 1, 14].

As a consequence, we established a Heine–Borel type theorem asserting that a subset of a finite-dimensional norm-compatible neutrosophic intuitionistic algebra fuzzy normed space is compact if and only if it is closed and bounded. This theorem provides an extension of finite-dimensional compactness results from fuzzy normed spaces to the present three-component setting [5, 4].

Finally, a Riesz-type separation lemma was formulated and used to characterize finite-dimensionality. More precisely, we proved that the compactness of a nontrivial closed neutrosophic intuitionistic algebra fuzzy ball implies that the underlying space is finite dimensional. Therefore, several fundamental principles of classical finite-dimensional functional analysis remain valid in the present framework, provided that suitable compatibility and regularity conditions are imposed.

The results also complement recent investigations of topological properties, continuous and bounded operators, Fréchet differentiation, and sequence spaces in neutrosophic normed spaces [11, 9, 10, 7]. Thus, the present work provides a basis for developing a broader theory of neutrosophic intuitionistic fuzzy functional analysis.

9. Suggestions for Further Research

The results established in this paper suggest several directions for future investigation.

(1) The theory of bounded and continuous linear operators between neutrosophic intuitionistic algebra fuzzy normed spaces should be developed. Earlier studies have established relationships between fuzzy continuity and fuzzy boundedness [2, 14], while continuous, bounded, and nonlinear operators have recently been investigated in neutrosophic normed spaces [9, 10]. It would be valuable to identify conditions under which continuity and boundedness are equivalent in the present algebraic framework.

(2) Compact, finite-rank, completely continuous, and Fredholm-type operators may be studied. In particular, one may investigate whether compact operator criteria based on bounded sequences, total boundedness, and convergent subsequences remain valid for all three neutrosophic components.

(3) The space of continuous linear functionals and the corresponding dual space should be introduced. Fuzzy dual spaces and fuzzy bounded functionals were considered by Bag and Samanta [2]. A neutrosophic intuitionistic fuzzy version of the Hahn–Banach theorem would therefore be a natural and important extension.

(4) The finite-dimensional norm-equivalence theorem may be investigated under weaker assumptions. More precisely, it would be useful to determine whether the norm-compatibility condition can be replaced by suitable homogeneity, radial monotonicity, convexity, or continuity assumptions. This problem is motivated by the established equivalence properties of finite-dimensional fuzzy normed spaces [5, 1].

(5) Compactness may be extended to infinite-dimensional settings. Possible topics include relative compactness, local compactness, weak compactness, weak sequential compactness, compact embeddings, and a neutrosophic intuitionistic fuzzy version of the Banach–Alaoglu theorem. Such work would extend existing investigations of compactness in fuzzy normed spaces [4].

(6) Open mapping, closed graph, and uniform boundedness principles may be formulated for complete neutrosophic intuitionistic algebra fuzzy normed spaces. Analogous principles have been considered in broader fuzzy functional-analysis frameworks [14]. Establishing these theorems would significantly strengthen the operator theory associated with the present spaces.

(7) Fréchet and Gâteaux differentiation of nonlinear mappings should be examined. Recent work on nonlinear operators between neutrosophic normed spaces provides a useful starting point [10]. Further research may investigate inverse and implicit function theorems in the three-component algebra fuzzy setting.

(8) The current framework may be generalized to neutrosophic intuitionistic algebra fuzzy 2-normed and n-normed spaces. Corresponding notions of linear dependence, convergence, completeness, compactness, and finite-dimensionality could then be developed. Related intuitionistic fuzzy 2-normed structures have been considered in [13].

(9) Different forms of generalized convergence may be introduced, including statistical convergence, ideal convergence, lacunary statistical convergence, rough convergence, weighted convergence, and matrix summability. Statistical convergence has already been investigated in intuitionistic fuzzy and neutrosophic normed spaces [6, 11], while Tribonacci-type convergence in neutrosophic normed sequence spaces was studied in [7].

(10) Fixed point theory provides another promising direction. Banach, Kannan, Chatterjea, Reich, and Ćirić type contractions may be formulated by controlling the truth-membership, indeterminacy-membership, and falsity-membership functions simultaneously. Existing fuzzy contraction principles and fuzzy continuous mappings may provide useful techniques for this development [14, 17].

(11) The dependence of the topology on the chosen continuous t-norm and t-conorm should be analyzed. In particular, one may characterize the properties that remain invariant

under a change of these operations and those that depend essentially on the selected product, minimum, probabilistic-sum, or maximum operations.

(12) Possible applications may be investigated in optimization, decision-making, control theory, differential and integral equations, artificial intelligence, and systems involving incomplete, inconsistent, or indeterminate information. Such applications could demonstrate how the three independent components provide additional modelling flexibility beyond ordinary fuzzy and intuitionistic fuzzy structures.

These problems may contribute to a systematic theory of neutrosophic intuitionistic fuzzy functional analysis and may reveal further connections among fuzzy algebra, topology, operator theory, approximation theory, and nonlinear analysis.

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